APPLICATIONS OF
MODELLING AND SIMULATION
http://arqiipubl.com/ams eISSN 2600-8084 VOL 3, NO. 1, 2019, 55-63

# Numerical Discretization Estimation for Ordinary Differential Equation via Hybrid Discretization

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*Received 18 November 2018, Revised 16 December 2018, Accepted 19 December 2018.*

**Abstract:** Simulation of control system is mostly developed based on the use of ordinary differential equation (ODE). With the advancement of technologies especially in term of real time computation, the traditional numerical approaches seem outdated to fit in to the current real-time discrete systems. Numerical method such as Euler's approach has inaccurate approximation as compared to other methods such as Heun's (RK2), Runge-Kutta (RK4), and Adams-Bashforth (AB2) methods, and these methods on the other hands suffers from high calculation time. In this work, Hybrid Discretization (HD) method is proposed to solve both approximation accuracy and calculation speed of the discretization. HD adapts RK2 method to correct the approximation error for one-to-ten step depending on the sampling time. Later, the system will return to Forward Euler's method to maintain the calculation speed. The HD is applied to two first order ODE test functions and the result of this work shows a significance improvement in terms of the accuracy of the approximation and slight improvement in term of the calculation time. The accuracy of about 9% is obtained as compared to a similar step time in Euler's, and comparable calculation time is maintained. In conclusion, it is shown that this new technique of discretization has better approximation than its counterparts and the method can serve as an important simulation tool in modeling and control of dynamic system.

Keywords: Hybrid Discretization; Numerical methods; Ordinary differential equation; Simulation.

## 1. INTRODUCTION

Ordinary differential equation (ODE), a function of one independent variable and its derivatives, plays an important part in system simulation, where the dynamic of system represented as a class of ODE is 'coded' into computing system so that the preliminary studies of the system can be performed. To solve ODE in the discrete time domain, a numerical method such as Euler's method is widely used where the solution can be tracked to represent the response of the system. Also, there are other numerical approaches such as Heun's (RK2), Runge-Kutta (RK4), and Adams-Bashforth (AB2) to solve the ODE problem [1-2] where each difference method results in different accuracy and computational calculation speed.

In many cases of simulation model of system in the form of ODE where Euler's approximation is being used to discretize the continuous ODE, the approach is simply taking the slope of the ODE function in order to predict the next sample. RK2 methods however in general will use the information of the slope at more than one sample point to extrapolate the solution to the future sample time. The AB2 method in contrary takes the integration between the present and next sample or alternatively accumulate the previous samples in integration. Based on this approach, AB2 method is also considered as multistep discretization method and due to the integral term, the calculation time is higher as compared to other methods. Moreover, the term calculation time is used as an indication for the time needed for the computer processor or microcontroller to evaluate single time step [3-4].

In this regards, Euler's method is more commonly used in various applications related to control. Specific to control system simulation, the discretization process will be used in the development of the system model and the control algorithm. The obvious choice of selecting Euler's method is due to the less number of iteration steps and this will be advantageous for the hardware implementation. For the control algorithm, the well-known proportional-integral-derivative (PID) which is commonly used in control system benefits from the simplicity of Euler's approach. However, for a higher order algorithm such as sliding mode control (SMC) and model predictive control (MPC), a more accurate discretization is usually more favorable [5-9]. MPC is highly dependent on the model of the system, thus selecting the proper discretization technique will totally affect the performance and accuracy of the MPC. Moreover, when MPC is discretized with Euler's and RK2 methods, it can be shown the difference in the accuracy, where the accuracy is higher with RK2 approach [10]. Similarly, in the work reported in [11-16] for SMC, Euler's method has been used with single-step techniques and the accuracy of the response is noticeably low. However, the response is improved for the higher order discretization approach. As the trade-off, the higher the order of the discretization process, the more time is needed to calculate and find the solution [17]. Algorithmic wise, Euler's method suffers from lower approximation accuracy due to one slope technique and this method cannot be suited for high-performance and high-speed applications when the accuracy and precision are needed.

The problem of selecting the discretization method in order to meet the application's requirement remains a challenging task in real time control. The need to have fast real-time response yet with high system accuracy needs to be balanced. Euler's fast response and other higher order discretization methods can be merged so that both of these performances can be achieved. Based on this motivation, a new hybrid discretization (HD) method based on optimal value Euler's method and multi-step discretization method is adapted. The merging of the two techniques is formed based on the performance assessed at each sample time of the discretized function. The accuracy of the response and the calculation times will be indexed at the end of the simulation of the response of the functions.

This paper is outlined as follows: Section 2 covers the reviews of the commonly used numerical techniques in modelling and simulation. Section 3 describes the analytical, the four existing the discretization and the newly proposed HD methods are run on two test functions and the errors in term of accuracy are shown. Section 4 shows a benchmark the proposed methods with other method in term of the calculation time under various sampling times. Finally, Section 5 includes the conclusion of this work.

## 2. HYBRID DISCRETIZATION APPROACH

ODE is a function that describes the relationship between dependent variables usually named as $y$ and $t$. The order of this function is determined from the highest order of the derivative terms. A linear ODE can be obtained when all the power of $y$ and its derivatives appearing in the ODE are non-negative integers and not exceeding unity [3]. The general form of ODE for a $n$-th order is:

$$a_n(t)\frac{d^n y}{dt^n} + a_{n-1}(t)\frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1(t)\frac{d\,y}{dt} + a_0(t)y = f(t) \tag{1}$$

Equation (1) is homogeneous if the condition $f(t) = 0$ is met. The derivative of the variable $y$ that is going to be discretized can be described in the following equation:

$$\frac{dy}{dt} = y' = f(t, y)\ ,\ \ y(t_0) = y_0 \tag{2}$$

where $y_0$ is the initial value for the function. In order to illustrate the HD method, the fundamental approach of the four existing discretization approaches is outlined as follows:

### 2.1 Euler's Discretization (FE)

The fundamental approach of different methods to calculate the approximation of ODE was first reported in 1768 by Leonhard FE [1-2]. The FE method has explicit (forward) and implicit (backward) for of discretization. There are differences between the two FE methods which it can reflect the stability of the system and the performance of the approximation. Yet the mainly used form of FE method in engineering applications is the forward method due to ease of implementation and the existence of explicitly solution. In FE approach, the slope is being used to extrapolate the next step (next sampling time). The forward FE form is based on Taylor series expansion and given as follows:

$$y_{n+1} = y_n + hf(y_n, t_n) \tag{3}$$

where $h$ represents the sampling time between the present and previous step as shown in Figure 1(a).

### 2.2 Heun's (RK2) Discretization

Heun's method (RK2) is also known as Predictor-Corrector method. The main property of this method is that it has the suitable combination of both implicit and explicit solution to ensure that the output has better convergence. It is a combination between the FE and AM2 (Adams-Moulton) methods and it has better performances than the FE method. The FE will be the predictor part which predicts the next step of the output, while the corrector part AM2 will correct the slope of the first output and this combination produces a bettercorrected output. The RK2 method can be described by the following equation:

$$y_{n+1}^p = y_n + hf(y_n, t_n) \qquad \text{(Predictor)} \tag{4}$$

$$y_{n+1} = y_n + \frac{h}{2}\left[f\left(y_{n+1}^p, t_{n+1}\right) + f(y_n, t_n)\right] \qquad \text{(Corrector)} \tag{5}$$

where $y_{n+1}^p$ is the output of the predictor function (4).

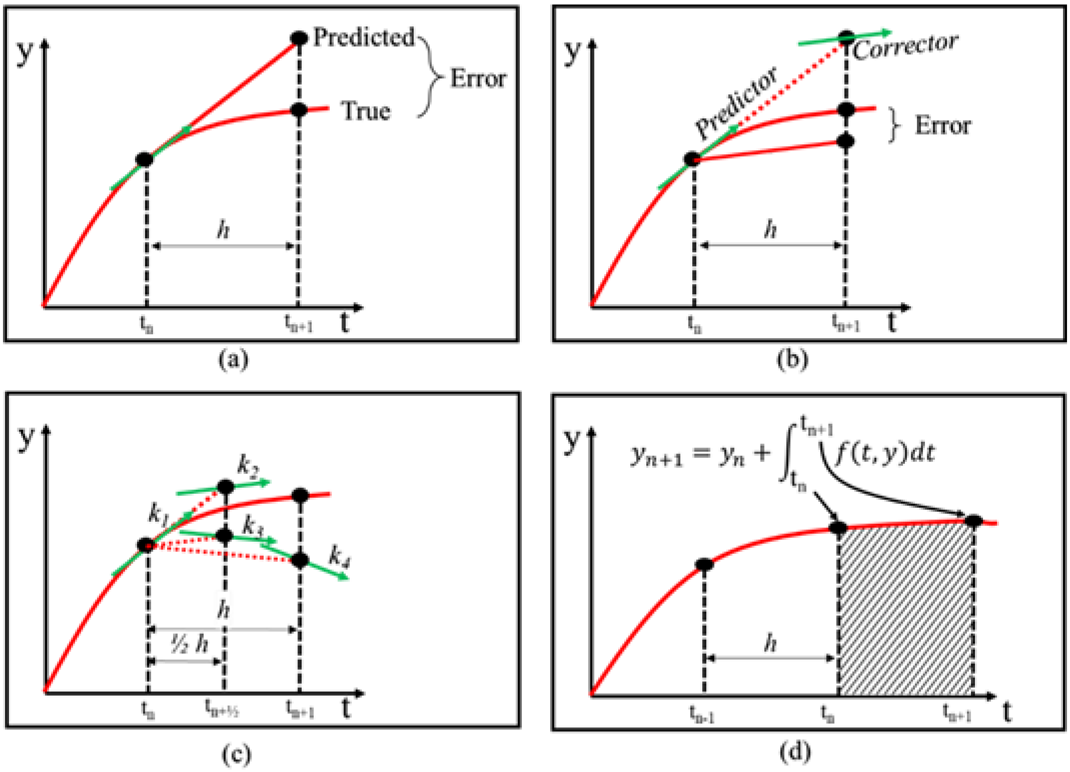


Figure 1. Illustration of numerical discretization methods: (a) FE, (b) RK2, (c) RK4, and (d) AB2

Figure 1(b) shows the application of RK2 method where *Predictor* and *Corrector* are the points of the function at each sample based on Equations (4) and (5). From this perspective, the RK2 method can also be considered as a hybrid method with switching factor is equal to one. This mechanism can be seen from Equations (4) and (5) where the the predictor 'measures' value and then the corrector update the values and this switching continues for the complete ODE function. It is obvious that the main challenge with this approach is the high effort for computation is required even the accuracy is improved. The reduced speed calculation due to the finding the solution is the main hindrance of adaption the method in real-time control system.

### 2.3 RK4 Discretization

Another method for approximation is the high order single-step method. This method has better convergence than the FE method because it uses four slopes to approximate instead of commonly used one or two slopes. There are several types of high order single-step methods reported and one of the most famous methods in multistep approach is the fourth order Runge-Kutta (RK4) method. In this method, four slopes are being used to approximate the future step and the following expressions describe the method:

$$k_1 = hf(y_n, t_n) \tag{6}$$

$$k_2 = hf(y_n + k_1/2, t_n + h/2) \tag{7}$$

$$k_3 = hf(y_n + k_2/2, t_n + h/2) \tag{8}$$

$$k_4 = hf(y_n + k_3, t_n + h) \tag{9}$$

$$y_{n+1} = y_n + (k_1 + 2k_2 + 2k_3 + k_4)/6 \tag{10}$$

It should be noted that if $k_2 = 0$, then $k_1$ will be half and the equation set will be similar to FE method. Figure 1(c) illustrates the approximation based on RK4.

### 2.4 Adam's Discretization

The next method that is also considered as multistep methods are the Adams method. Adams method concept is performed by integrating the slope of the function between the time intervals $(t_n, t_{n+1})$. This approximation is considered among the most precise technique in terms of accuracy due to slope integration. Adams method has two different types which are Adam-Bashforth (AB2) with explicit solution while the implicit one is called Adams-Moulton (AM2). Both AB2 and AM2 are second-order approximation where AB2 needs less calculation as compared with AM2, but AM2 is more stable in term of performance. First order Adams is simply FE method and AB2 can be given in the following equation:

$$y_{n+1} = y_n + \frac{3h}{2} f\big(y_n, t_n - f(y_{n-1}, t_{n-1})\big) \tag{11}$$

The main property in Adams method is the integration of the slope to have smooth and more accurate approximation as shown in Figure 1(d).

### 2.5 HD Discretization

Based on the aforementioned methods, the hybrid discretization (HD) method been proposed to solve the problem of approximation and calculation time. The main idea of HD is to use the same RK2 method and FE methods in combination with switching mode such that the system will apply FE method for the first step and then the RK2 is executed to do the correction for the FE approximation.

The HD algorithm can be described by the following algorithm and also shown in Figure 2. The algorithm will start in step 1 with considering the initial value of 1. Subsequently, in step 2 the solution will be exactly like the FE method in Figure 1(a). In step 3, RK2 will be evaluated for $k$ times and this step will have the accuracy correction to achieve enhanced results as illustrated in Figure 3. Finally, step 4 will return to conventional FE method exactly like step 1 and continue to the end of the approximation.

Step 1: Start at ($n=0$) Assumption of the initial value been made $y_0 = 1$.
Step 2:($n=1$) Solving the ODE in FE method for one step (3).
Step 3:($n=2\ldots\ n=k$) Solving the ODE in RK2 method for $k$ steps (4) and (5).
Step 4:($n=k+1\ldots n=\infty$) Finally, the ODE will return to be solved in FE method for the end of the ODE (3).

$k$ is the number of steps that the RK2 method will be evaluated. The value of the factor $k$ can be (1, 2, …, $n$), and $n$ is any real number. This value could be depended on the ODE complexity, moreover, using HD with $k$=1, better accuracy will be obtained while maintaining the speed of calculation. On the in the other hand, if $k$ >2 then higher accuracy approximation will be obtained and concurrently the calculation time will be increased proportionally. Therefore, the value of $k$ is selected based on the need of the application while achieving the requirement of the system. The illustration of the HD approximation is shown in Figure 3 where it can be noticed that the approximation error based on HD is significantly reduced as compared with FE approach.

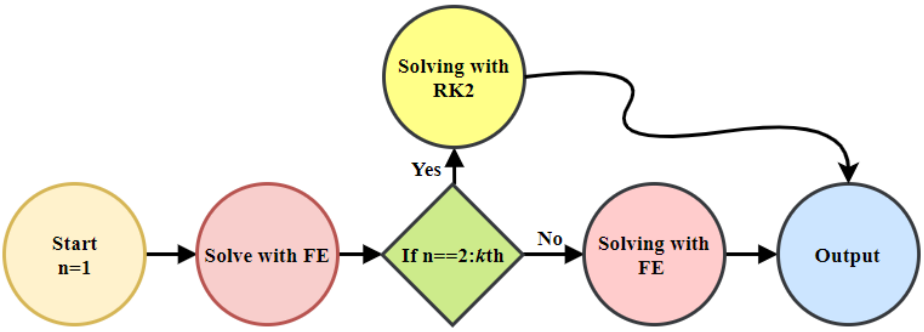


Figure 2. The flow of the HD algorithm

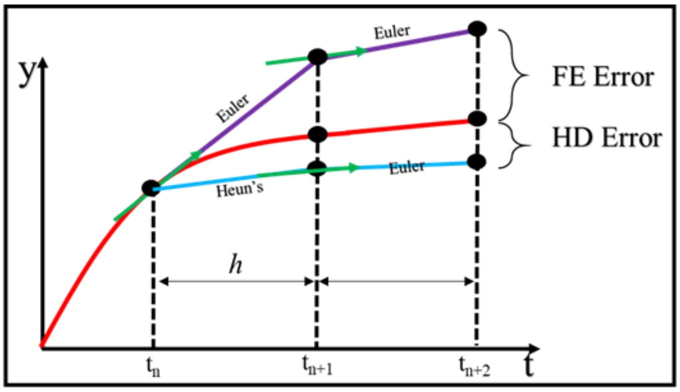


Figure 3. Illustration of HD algorithm as compared to FE

## 3. NUMERICAL TEST FUNCTION

To test the aforementioned methods two test functions will be used to compare the methods and study their performance. The first function is the natural exponential $e^t$ with respect to time $t$ to be tested with different methods and sampling times, and this example can measure the convergence of the tested numerical method [3]:

$$\frac{dy(t)}{dt} = y(t) \tag{12}$$

The responses of this ODE with different time steps and different methods are shown in Figure 4. It can be shown that the response of the various methods are different with the approximation methods. The new proposed method is shown to improve the error ratio with reduced calculation time as compared to other multistep methods.

Comparing the proposed HD method with the FE method at different sampling times (1, 0.5, 0.1, 0.05, 0.01, and 0.005 sec), it is shown that a significant enhancement is achieved as shown in Figure 5. It is observed that the HD can achieve high accuracy at a sampling time of 0.01 sec while the FE needs a sampling time of 0.005 sec to achieve a similar accuracy level. This response is going to close proximity of half-step in the previous curve and with very good accuracy and the calculation time is maintained.

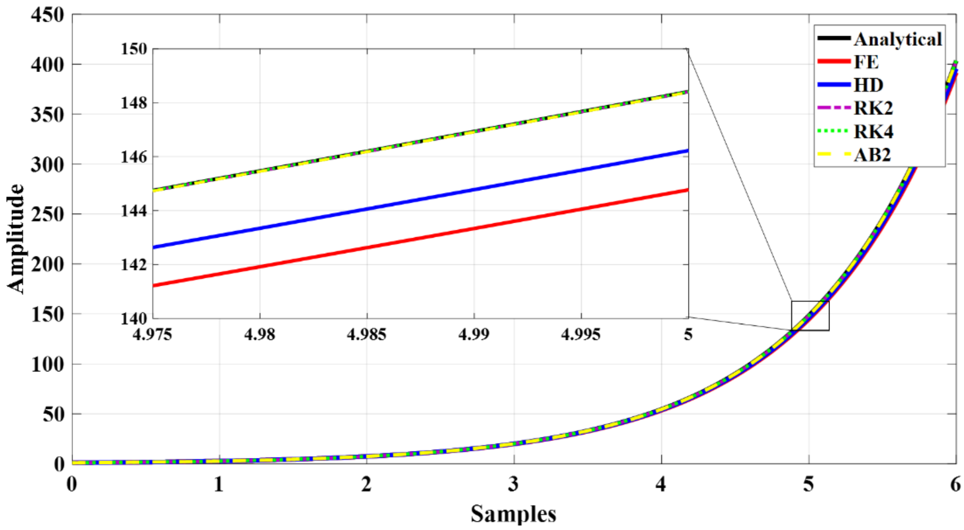


Figure 4. Numerical solutions of different methods and the analytical solution of $e^t$ with sampling time of 0.01 sec

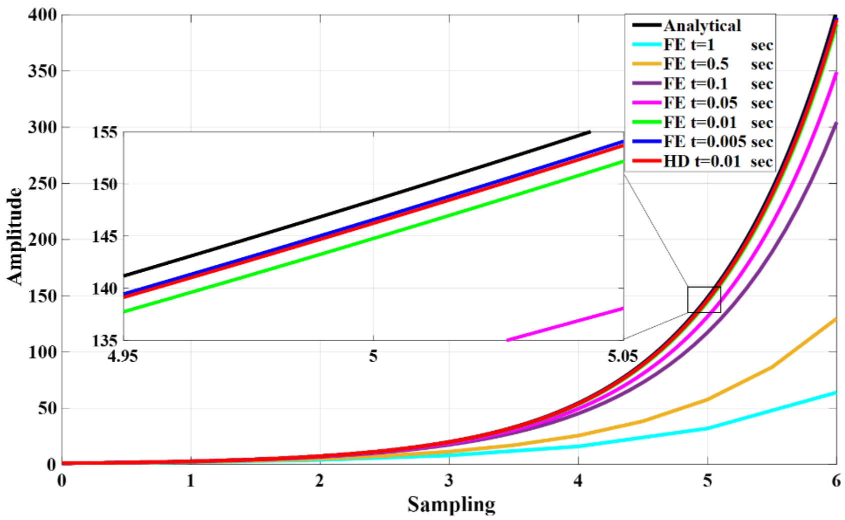


Figure 5. Numerical solutions with FE method with different step sizes, HD (0.01 sec) method, and analytical solution for $e^t$

The second example that will illustrate the performance of the HD methods is the ODE function given as:

$$\frac{dy}{dt} = -2t^3 + 12t^2 - 20t + 8.5 \tag{13}$$

Equation (13) is used to test the numerical methods when having changing in slope from negative and positive and vice versa, and that can test the convergence of the tested numerical method [3]. Figure 6 shows the different methods with the same sampling time. It can be noticed that the proposed HD method is better than FE method however the accuracy is less as compared to other methods. In Figure 7 shows a comparison of different sampling time of FE method and 0.01 sec sampling time of the proposed HD. As shown previously, HD method gives better approximation as compared to FE method.

Both of the previous two examples described the accuracy of the ODE with respect to the sampling time. To measure the significance of the accuracy of the proposed method, Figure 8 and Figure 9 show the Root Mean Square Error (RMS) of ODE function 1 and ODE function 2 respectively under various sampling times. These measurements show that the proposed method has a better accuracy with 9% with both examples as compared with conventional FE method. From these figures, it can be observed that the RMS value for HD method is slightly lower than FE method at all sampling times but much higher than other methods. As the sampling times increase, the RMS values for all the methods increase proportionally as expected due to the error in the approximation.

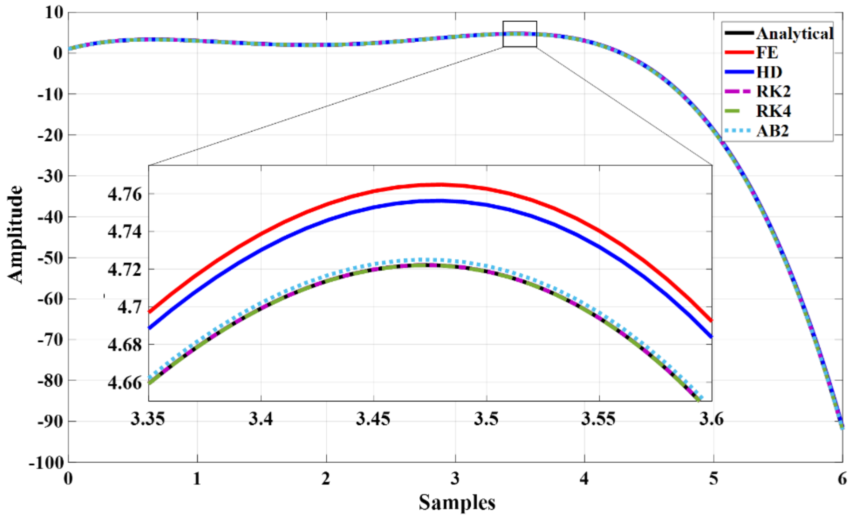


Figure 6. Numerical solutions with different methods and the analytical solution of the second function with sampling time of 0.01 second

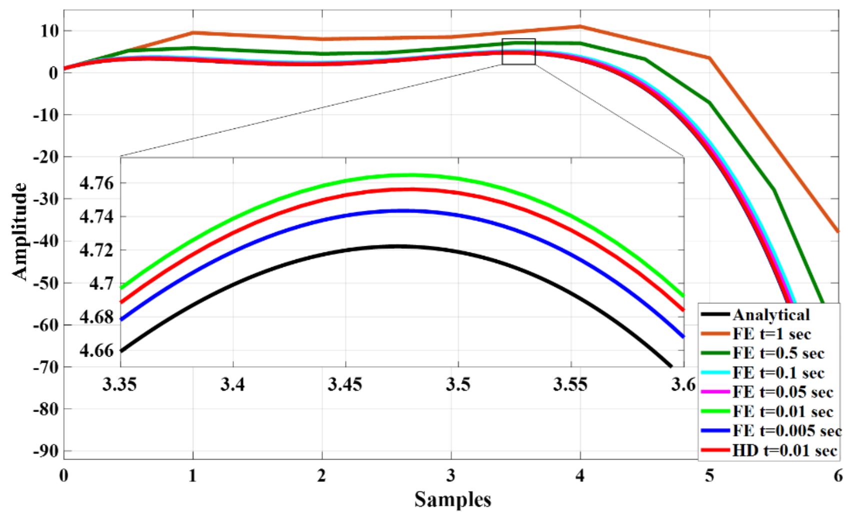


Figure 7. Numerical solutions with FE method with different step sizes, HD (0.01 sec) method, and analytical solution for the second function

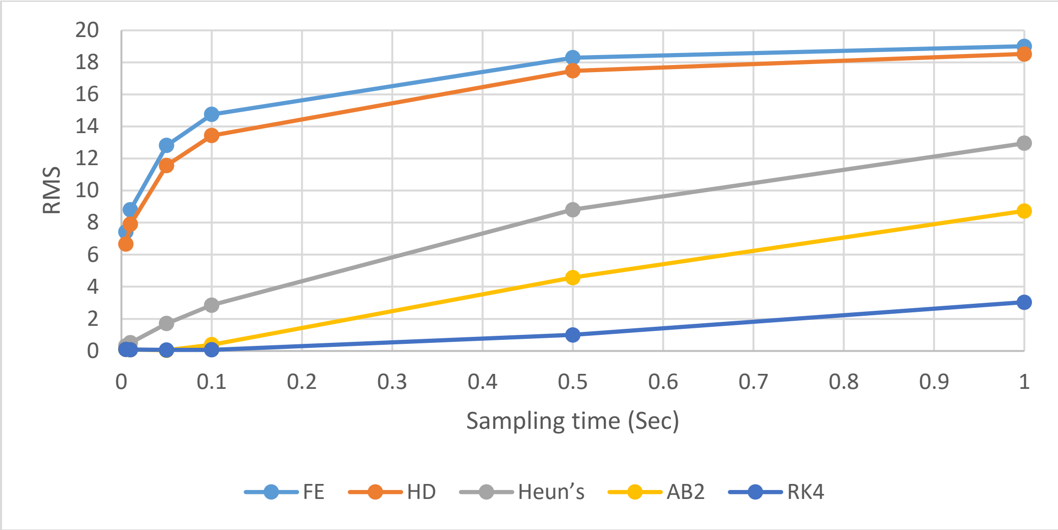


Figure 8. The RMS of all methods with different sampling time for Function 1

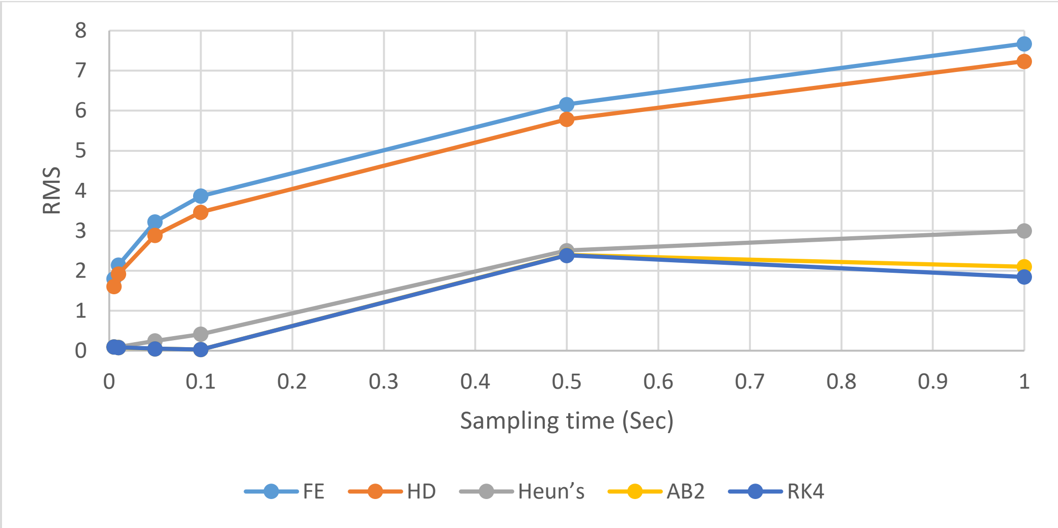


Figure 9. The RMS of all methods with different sampling times Function 2

## 4. SIMULATION CALCULATION TIME TEST

For this test, Matlab m-file has been used to calculate the calculation time of the two test functions of all the methods at the set samplings time. This test has been implemented on a computer (core i7 720qm 1.6 GHz quad-core, RAM 8 GB, at room temperature) with real-time priority configuration is setup to make sure the results will be in the range of the processor speed and to guarantee the speed is consistent. The speed is measured by using a tic-toc function with $10^6$ iterations for each method.

Figure 10 and Figure 11 show the calculation time for each method. From both charts it is obvious that the calculation time for HD method is almost the same as the FE method approximation and this reveals that in compression between the error and time calculation, the HD is better than FE method and can be implemented on systems instead of FE method due to better accuracy and with the same calculation time. Noticing that in smaller step sizes the calculation time becomes larger exponentially and that is due to higher calculation per step. In other word, while this is real-time calculation the computer needs more time to execute single step per processor clock. In this test, the result of the AB2 method was not included as it consumes too much time more than 30 seconds to solve a single equation, that's due to the integration term in the AB2 method. Finally, the accuracy of the solution of the numerical methods is highly proportional to the time needed from the processor to solve an ODE, and by the proposed method, a moderate combination for both calculation time and the accuracy of the approximation can be achieved that can be used in real-time simulation applications.

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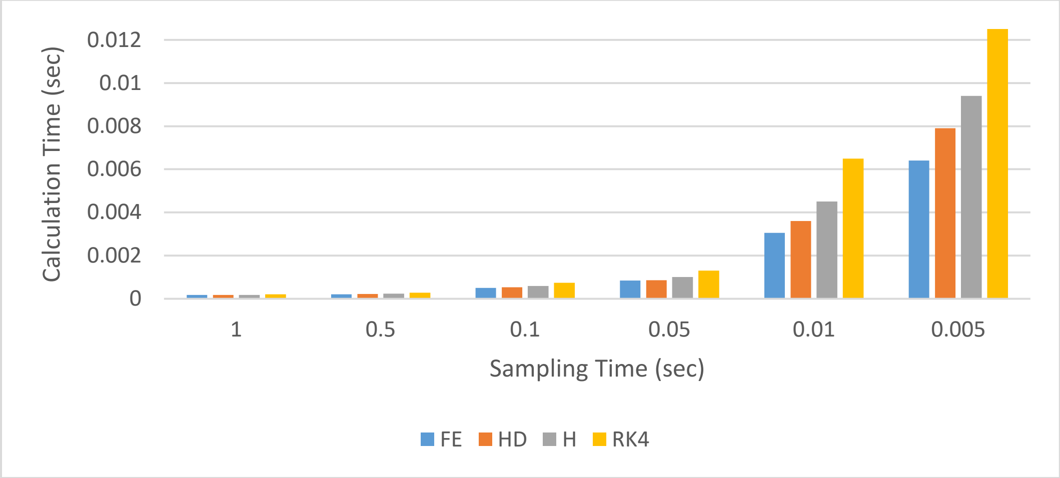


Figure 10. The calculation time of all methods with different sampling times for Function 1

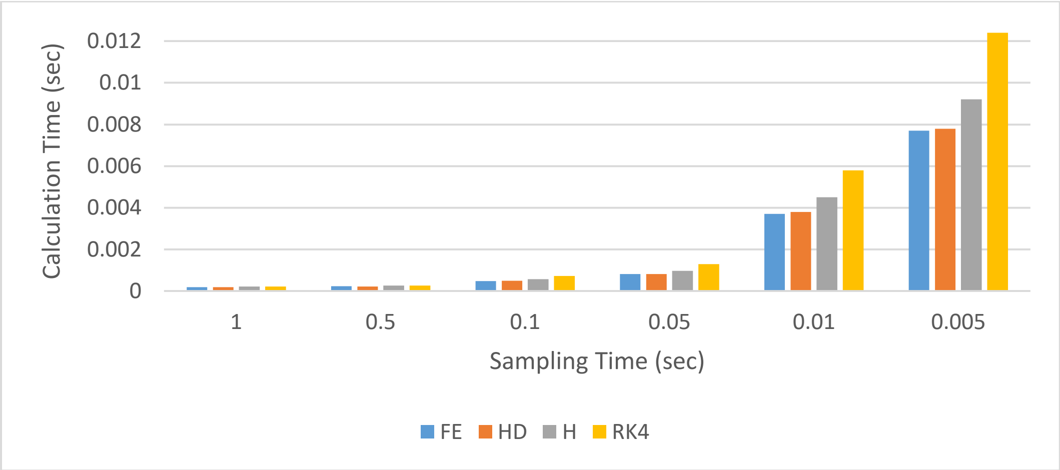


Figure 11. The calculation time of all methods with different sampling times for Function 2

## 5. CONCLUSION

ODE needs to be discretized with a numerical method, and this can be done by choosing one of the numerical methods. FE method has a fast response yet it suffers inaccurate approximation and the other high-level methods need high calculation time. The new HD method has been proposed to solve the problem and to reconcile between the high speed and good accuracy and that has been tested with two different first order ODE functions. Good results has been achieved in both calculation time and accuracy as compared to the other method. Further implementation on discretization of close-loop control can be performed to test the feasibility of the HD method when implemented in feedback control system.

## REFERENCES

[1] H. Euler, Institutiones calculi integralis. Volumen Primum, Opera Omnia, vol. XI, B. G. Teubneri Lipsiae et Berolini MCMXIII, 1768.
[2] C. Runge, Ueber die numerische auflsung von differentialgleichungen, *Mathematische Annalen*, 46(2), 167–178, 1895.
[3] S. C. Chapra and R. P. Canale, *Numerical Methods for Engineers*, 7th Edition. McGraw-Hill Science/Engineering/Math, 2015.
[4] D. Griffiths and D. J. Higham, *Numerical Methods for Ordinary Differential Equations: Initial Value Problems*. Springer, 2010.
[5] S. Bolognani, S. Bolognani, L. Peretti and M. Zigliotto, Design and implementation of model predictive control for electrical motor drives, *IEEE Transactions on Industrial Electronics*, 56(6), 1925–1936, 2009.
[6] S. Vazquez, J. Rodriguez, M. Rivera, L. G. Franquelo and M. Norambuena, Model predictive control for power converters and drives: Advances and trends, *IEEE Transactions on Industrial Electronics*, 64(2), 935–947, 2017.
[7] K. Kiran, S. Das and D. Singh, Model predictive field oriented speed control of brushless doubly-fed reluctance motor drive, *International Conference on Power, Instrumentation, Control and Computing*, Thrissur, India, 2018, pp. 1–6.

[8] I. G.-Prieto, M. J. Duran, J. J. Aciego, C. Martin and F. Barrero, Model predictive control of six-phase induction motor drives using virtual voltage vectors, *IEEE Transactions on Industrial Electronics*, 65(1), 27–37, 2018.
[9] S. Nalakath, M. Preindl and A. Emadi, Online multi-parameter estimation of interior permanent magnet motor drives with finite control set model predictive control, *IET Electric Power Applications*, 11(5), 944–951, 2017.
[10] Y. Zhang, Y. Bai and H. Yang, A universal multiple-vector-based model predictive control of induction motor drives, *IEEE Transactions on Power Electronics*, 33(8), 6957–6969, 2018.
[11] S. Li, H. Du and X. Yu, Discrete-time terminal sliding mode control systems based on Euler's discretization, *IEEE Transactions on Automatic Control*, 59(2), 546–552, 2014.
[12] H. Du, X. Chen, G. Wen, X. Yu and J. Lu, Discrete-time fast terminal sliding mode control for permanent magnet linear motor, *IEEE Transactions on Industrial Electronics*, 65(12), 9916–9927, 2018.
[13] G. Sun, Z. Ma and J. Yu, Discrete-time fractional order terminal sliding mode tracking control for linear motor, *IEEE Transactions on Industrial Electronics*, 65(4), 3386–3394, 2018.
[14] Z. Kuang, Z. Yu, G. Sun, W. Lin and H. Gao, Contouring control of linear motor direct-drive X-Y table via chattering-free discrete-time sliding mode control, *IEEE 27th International Symposium on Industrial Electronics*, Cairns, Australia, 2018, pp. 201–205.
[15] Y. Lu and J. Zhao, A sliding mode flux observer for predictive torque controlled induction motor drive, *The 30th Chinese Control and Decision Conference*, Shenyang, China, 2018, pp. 3280–3285.
[16] S. Li, H. Du and X. Yu, Discrete-time terminal sliding mode control systems based on euler’s discretization, *IEEE Transactions on Automatic Control*, 59(2), 546–552, 2014.
[17] L. Dieci and L. Lopez, Numerical solution of discontinuous differential systems: Approaching the discontinuity surface from one side, *Applied Numerical Mathematics*, 67, 98–110, 2013.