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Template Method for Exact Value Calculation of Root Locus Break Points

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Abstract: One of the methods used to examine stability, to analyze, and to design a linear time invariant control system for a single input and a single output is the Root Locus technique. It is known that Root Locus technique is a powerful and efficient mean used in control of the systems. In addition, it determines the range of gain for a specific type of the time response of a control system. There are several important points on the root locus graph along the real axis of the *s*-plane. These points are known as the Breakaway, and the Break-in points. In this article those points are called Break points, and the polynomial that some of its roots Break points is called Break polynomial. In this article another method is presented for obtaining the Break polynomial. This method is called the Template method. The mathematical proof of the correctness of the basis of the method is presented. In this method the Break polynomial is obtained by programing or by hand calculation as with other common methods. The hand calculation's part is presented only. The merit of this method is that it is applicable for any system's order, and it can handle complex poles and complex zeros. It is shown that the technique of the template filling is a systematic procedure. So that if a template is used for a specific order of control system it can be expanded to handle higher order system just by adding another row on the top of the existing template. The Template method can be computer programmed and if it is incorporated within the MATLAB root locus graphs program it enhances the graph by presenting the exact values of the Break points on it. The method is compared with other common methods in the solution of examples of control systems to show its simplicity for users, and to show its correctness for its capability of handling various orders of a control systems. The results showed that the method gives accurate results and the required number of mathematical operations of the calculations to obtain the result is about 30% of the mathematical operations required by other methods.

Keywords: Break-in point calculation; Breakaway points' calculation; Break polynomial's calculation; Gain at breakaway point; Template method for break points' calculation.

1. INTRODUCTION

The root locus technique is a semi graphical method. This method introduced by Evans in years 1948 [1] and 1950 [2]. This technique is based on plotting the locus of the roots of the closed-loop characteristic equation on a complex plane, *s*-plane. The roots are the closed loop poles. The graph of the locus of the roots on the complex *s*-plane for varying a parameter of interest is known as the root loci graphs. Usually, the static gain of the open loop system is the parameter of interest. The static gain various from zero to infinity. The time response of a linear control system is the sum of all responses that are contributed by its roots. The real root gives exponential response, while a complex conjugate pair gives oscillatory response [3, 4, 5]. The static gain K value in root locus technique defines the location of the system roots on the *s*-plane, and consequently the response components' shape. The gain at the Break points is a border value for some roots, and any increase above this value changes the roots' type.

As a result of value change of the static gain K from zero to infinity the root loci move from poles toward zeros. As a result of this movement two cases may occur. One case that two root loci leave two poles on the real axis of the *s*-plane and move in opposite directions. This leads to collision at a point. This point is called a Breakaway point, and a second case when a pair of complex conjugate roots move in opposite directions toward the real axis they will collide at a point on the real axis. This point is called Break-in point. Algebraically the characteristic equation has double roots at a Break point, and geometrically, the Break points are located on the root locus segments between two poles, or between two zeros.

Larger value of K than its value at the Breakaway points the characteristic equation has at least one pair of complex conjugate roots, and consequently the two loci split and leave the real axis in opposite direction and continue to move in a symmetric fashion, while for a larger value of K than its value at the Break-in point the characteristic equation has two real

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roots, and the root loci split and continue to move at different directions on the real axis. The process of the Break-in point is the opposite process of the Breakaway point.

In literature there are several methods for finding the Break points and the corresponding gains [3, 6, 7]. The common used method is based on solving K in the characteristic equation to have a fraction of two polynomials, and then finding the local extremum for K. The extremum is found via differentiation of the mathematical expression of K with respect to the variable s and setting the derivate to zero. The result is an algebraic polynomial. Some roots of this polynomial are the Break points. There is a need to continue the search for the proper roots which could be a Break points. The condition for a root to be considered as Break point is to be on the root loci segments which are on the real axis and satisfy the angle condition. In this via Differentiation method, the mathematical differentiation difficulty increases when degree of the numerator's polynomial is higher.

A second method was presented by Remec [8]. It is a tabulation method. This method is resemblance to the construction of the Routh-Hurwitz array, and it is a long process. A third method was presented by Franklin [9] and is called the via Transition method. This method is based on that the natural logarithm has a zero derivative at the same point that the parameter of interest's derivative expression is zero. It does not require differentiation, but there are several polynomials' factors multiplications to obtain the final expression of the Break polynomial. A fourth method was developed by Krishnan [9]. This method is based on a successive differentiation of numerator and denominator to construct arrays to obtain the algorithm, and it is a long process. A fifth method named the Formulated method was developed by Shibly [6]. In this method the Break polynomial is obtained in a formulated approach.

2. THEORETICAL BACKGROUND

A common block diagram description of a single input and a single output of a linear invariant control systems is shown in Figure 1. The open loop transfer function of this control system is $G(s)$, and its feedback transfer function is $H(s)$. The general form of its closed-loop transfer function $T(s)$ and has negative feedback, is

$$T(s) = \frac{G(s)}{1 + G(s)H(s)} \tag{1}$$

To obtain the characteristic equation (C.E), $\Delta(s)$, of the control system is by setting the denominator of the closed-loop transfer function $T(s)$, Equation (1), to zero to have:

$$\Delta(s) = 1 + G(s)H(s) = 0 \tag{2}$$

The locus of the roots of the characteristic equation of the closed loop control system Δ(s), Equation (2), gives the Root Locus graph. The transfer function $G(s)H(s)$ has a rational form of two polynomials such as

$$G(s)H(s) = K\frac{N(s)}{D(s)} \tag{3}$$

The degree of the denominator D(s) is n, and its roots are called poles, while the degree of the numerator N(s) is m, and its roots are called zeros. In general, $m \leq n$. Those polynomials can be written as summation of terms as follows:

$$D(s) = a_0 + \sum_{q=1}^{n} a_q s^q\,, \qquad N(s) = b_0 + \sum_{q=1}^{m} b_q s^q \tag{4}$$

Substitute Equation (3) into Equation (2) to obtain the characteristic equation as

$$\Delta(s) = 1 + K\frac{N(s)}{D(s)} = 0 \tag{5}$$

Simplify Equation (6) to find that the numerator is equal to zero to have

$$\Delta(s) = D(s) + KN(s) = 0 \tag{6}$$

The parameter K theoretically varies from zero to infinity. As a result, the characteristic equation, Equation (6), has n new roots for each new value of K. The locus of those roots is the root locus graph. At the two limit values of K the characteristic equation becomes

$$\text{For}\quad K = 0 \quad \Delta(s) = D(s) = 0, \ \text{and for} \quad K \to \infty \quad \Delta(s) = N(s) = 0, \tag{7}$$

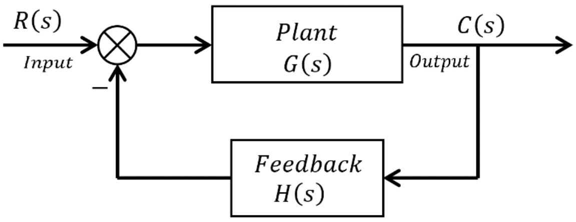


Figure 1. Feedback control system block diagram

Equation (7) shows that the roots of the characteristic equation for $K = 0$ are the roots of $D(s)$, the poles of the open loop transfer function, while when $K \to \infty$ the roots are the roots of the nominator $N(s)$, the zeros of the open loop transfer function. This means that the root loci graph starts at the poles and ends at the zeros of the open loop transfer function. As a result, the number of root loci is equal to the number of poles. The points on the root locus graph must satisfies two conditions: the magnitude condition, and the angle condition. The real roots of the characteristic equation define the root locus graph's segments on the real axis. To satisfy the angle condition of the root locus segments on the real axis must be located to the left of an odd number of the sum of poles and zeros. While the conjugate pair roots of the characteristic equation define the symmetrical root loci segments about the real axis.

In the fifth method [6] the Break polynomial is obtained in a formulated approach, and the equation of the Break polynomial is

$$P_f(\sigma) = P_1(\sigma)P_4(\sigma) - P_2(\sigma)P_3(\sigma) = 0 \quad (8)$$

Equation (8) consists of four polynomials P_i for $i = 1, \cdots, 4$. They are constructed from the denominator polynomial's coefficients, a_i, and from the coefficients b_j of the numerator's polynomial of the open loop transfer function. Those polynomials are:

$$P_1(\sigma) = \sum_{q=1}^{n} (-1)^q q a_q \sigma^{q-1} \quad (9)$$

$$P_1(\sigma) = \sum_{q=1}^{m} (-1)^q q a_q \sigma^{q-1} \quad (10)$$

$$P_3(\sigma) = a_0 + \sum_{q=2}^{n} (-1)^{q-1} (q-1) a_q \sigma^q \quad (11)$$

$$P_4(\sigma) = b_0 + \sum_{q=2}^{m} (-1)^{q-1} (q-1) b_q \sigma^q \quad (12)$$

Some of the roots (sigma values) of the Break polynomial, Equation (8), are real. The real roots are located on the real axis but not all real roots are Break points. For a real root to be a Break point is to be on the segment which is to the left of an odd number of the sum of poles and zeros of the system open loop transfer function. By replacing the independent variable sigma by negative s,$(\sigma = -s)$, in the first polynomial P_1, Equations (9), to obtain

$$P_1(s) = \sum_{q=1}^{n} (-1)^q q a_q (-s)^{q-1} = \sum_{q=1}^{n} (-1)^q (-1)^{q-1} q a_q (s)^{q-1} \quad (13)$$

The product of the two exponent factors of the number negative ones in Equation (13) is simplified to give

$$(-1)^q(-1)^{q-1} = (-1)^{q-1}(-1)^q = -1 \quad (14)$$

Substitution of Equation (14) into Equation (13), to have the polynomial $P_1(s)$ as function of s gives new form is

$$P_1(s) = -\sum_{q=1}^{n} q a_q s^{q-1} \quad (15)$$

Repeat the same simplification for the other three polynomials as in Equations (15) to have them as function of s and they are

$$P_2(s) = -\sum_{q=1}^{n} q b_q s^{q-1} \quad (16)$$

$$P_3(s) = a_0 - \sum_{q=1}^{n} (q-1) a_q s^q \quad (17)$$

$$P_4(s) = b_0 - \sum_{q=1}^{n} (q-1) b_q s^q \tag{18}$$

Substitution of Equations (15-18) into the Break polynomial, Equation (8) gives the Break polynomial as function of s such as

$$P(s) = -\left(\sum_{q=1}^{n} q a_q s^{q-1}\right)\left(b_0 - \sum_{q=1}^{n} (q-1) b_q s^q\right) + \left(\sum_{q=1}^{n} q b_q s^{q-1}\right)\left(a_0 - \sum_{q=1}^{n} (q-1) a_q s^q\right) \tag{19}$$

This article is divided into two parts. The first part presents a mathematical proof of the correctness of the basics of the proposed new template method for calculating the Break points and the corresponding gain. The second part presents the development of the template method, and comparison with another used methods through example solving, and analysis of its computational efficiency. One of the used methods for determining the Break points is the Franklin via Transition method. This method is based on using the below formula to obtain the Break polynomial. This method has no differentiation during the algebraic operations. Therefore, it is more favorable for the users who are not comfortable with a differentiation of a fractional form of the static gain's expression of the control system. A factorial form of the considered transfer function is

$$G(s)H(s) = K \frac{\prod_{j=1}^{m} (s + z_j)}{\prod_{i=1}^{n} (s + p_i)} \tag{20}$$

The method formula is based on the transfer function form given in Equation (20).

$$\sum_{i=1}^{n} \frac{1}{s + p_i} = \sum_{j=1}^{m} \frac{1}{s + z_j} \tag{21}$$

To demonstrate the use of this formula, Equation (21), is to expand it as an addition of fractions such as

$$\frac{1}{s+p_1} + \frac{1}{s+p_2} + \frac{1}{s+p_3} + \cdots + \frac{1}{s+p_n} = \frac{1}{s+z_1} + \frac{1}{s+z_2} + \frac{1}{s+z_3} + \cdots + \frac{1}{s+z_m} \tag{22}$$

The common denominator of the fractions in Equation (22) is the product of all fractions' denominators. Adding the fractions in both sides to have

$$\frac{[(s+p_2)(s+p_3)\cdots(s+p_n)] + [(s+p_1)(s+p_3)\cdots(s+p_n)] + [(s+p_1)(s+p_2)\cdots(s+p_n)] + [(s+p_1)(s+p_2)(s+p_3)\cdots]}{(s+p_1)(s+p_2)(s+p_3)\cdots(s+p_n)}$$
$$\frac{[(s+z_2)(s+z_3)\cdots(s+z_m)] + [(s+z_1)(s+z_3)\cdots(s+z_m)] + [(s+z_1)(s+z_2)\cdots(s+z_m)] + [(s+z_1)(s+z_2)(s+z_3)\cdots]}{(s+z_1)(s+z_2)(s+z_3)\cdots(s+z_m)} \tag{23}$$

It can be noticed that the numerator is the derivative of the denominator in both sides of the equation. As a result, Equation (23) may be written as

$$\frac{D'}{D} = \frac{N'}{N} \tag{24}$$

or

$$-D'N + N'D = 0 \tag{25}$$

Differentiate both polynomials $D(s)$, and $N(s)$, Equation (4), to have

$$D'(s) = \sum_{q=1}^{n} q a_q s\sigma^{q-1} \tag{26}$$

$$N'(s) = \sum_{q=1}^{m} q b_q \sigma^{q-1} \tag{27}$$

Substitute the polynomials' derivatives, Equations (26-27), into Equation (25) to have the Break polynomial P_F of this method. The degree of this polynomial is $(n + m - 1)$.

$$P_F(s) = \left(\sum_{q=1}^{m} q b_q s^{q-1}\right)\left(a_0 + \sum_{q=1}^{n} a_q s^q\right) - \left(\sum_{q=1}^{n} q a_q s^{q-1}\right)\left(b_0 + \sum_{q=1}^{m} s_q \sigma^q\right) \tag{28}$$

2.1 The correctness of the Template Method's Break Polynomial

The correctness of the bases of the Template method is proved by showing that the Template method's Break polynomial is the same polynomial that achieved by other method. This means that $P(s)$, the Break polynomial of the Template method, is the same polynomial $P_F(s)$ of the Via Transition method. This is achieved by showing that both sides of the following equation are the same. See Appendix A for proof.

$$P(s) = P_F(s) \tag{29}$$

2.2 Derivation of the Template Method

The derivation of the Template method is shown by finding the Break polynomial of two examples of a control systems. The first example is for fourth order system, and a second example for five order system. Then showing how the template may be modified to be applicable for a higher order system. The modification is shown for the modification of template of fourth order system to a template of fifth order system. The polynomials, $P_i(s)$, of a four-order system are

$$P_1(s) = -\sum_{q=1}^{n} q a_q s^{q-1} = -(a_1 + 2a_2 s + 3a_3 s^2 + 4a_4 s^3) \tag{30}$$

$$P_2(s) = -\sum_{q=1}^{n} q b_q s^{q-1} = -(b_1 + 2b_2 s + 3b_3 s^2 + 4b_4 s^3) \tag{31}$$

$$P_3(s) = a_0 - \sum_{q=1}^{n} (q-1) a_q s^{q} = a_0 - (a_2 s^2 + 2a_3 s^3 + 3a_4 s^4) \tag{32}$$

$$P_4(s) = b_0 - \sum_{q=1}^{n} (q-1) b_q s^{q} = b_0 - (b_2 s^2 + 2b_3 s^3 + 3b_4 s^4) \tag{33}$$

Substitution of the four polynomials, Equations (30-33), into the Break polynomial, Equation (8), and performing the algebraic multiplication and then reordering the terms to obtain the Break polynomial $P(s)$ having the following form:

$$\begin{aligned} P(s) = (a_4 b_3 - a_3 b_4) s^6 + 2(a_4 b_2 - a_2 b_4) s^5 + [3(a_4 b_1 - a_1 b_4) + (a_3 b_2 - a_2 b_3)] s^4 +, \\ +[4(a_4 b_0 - a_0 b_4) + 2(a_3 b_1 - a_1 b_3)] s^3 + [3(a_3 b_0 - a_0 b_3) + (a_2 b_1 - a_1 b_2)] s^2 +, \\ +2(a_2 b_0 - a_0 b_2) s + (a_1 b_0 - a_0 b_1) = 0 \end{aligned} \tag{34}$$

The Break polynomial, Equation (34), can be divided into four parts as follows:

$$\begin{aligned} &\text{1st:} \quad (a_4 b_3 - a_3 b_4)(1s^6) + (a_4 b_2 - a_2 b_4)(2s^5) + (a_4 b_1 - a_1 b_4)(3s^4) + (a_4 b_0 - a_0 b_4)(4s^3) \\ &\text{2nd:} \quad (a_3 b_2 - a_2 b_3)(1s^4) + (a_3 b_1 - a_1 b_3)(2s^3) + (a_3 b_0 - a_0 b_3)(3s^2) \\ &\text{3rd:} \quad (a_2 b_1 - a_1 b_2)(1s^2) + (a_2 b_0 - a_0 b_2)(2s) \\ &\text{4th:} \quad (a_1 b_0 - a_0 b_1)(1s^0) \end{aligned} \tag{35}$$

The four parts of Equation (35) may be written in the form of determinants and factors as follows:

$$\begin{aligned} &\text{1st:} \quad \begin{vmatrix} a_1 & a_0 \\ b_1 & b_0 \end{vmatrix} (1s^0) \\ &\text{2nd:} \quad \begin{vmatrix} a_2 & a_1 \\ b_2 & b_1 \end{vmatrix} (1s^2) + \begin{vmatrix} a_2 & a_0 \\ b_2 & b_0 \end{vmatrix} (2s^1) \\ &\text{3rd:} \quad \begin{vmatrix} a_3 & a_2 \\ b_3 & b_2 \end{vmatrix} (1s^4) + \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix} (2s^3) + \begin{vmatrix} a_3 & a_0 \\ b_3 & b_0 \end{vmatrix} (3s^2) \\ &\text{4th:} \quad \begin{vmatrix} a_4 & a_3 \\ b_4 & b_3 \end{vmatrix} (1s^6) + \begin{vmatrix} a_4 & a_2 \\ b_4 & b_2 \end{vmatrix} (2s^5) + \begin{vmatrix} a_4 & a_1 \\ b_4 & b_1 \end{vmatrix} (3s^4) + \begin{vmatrix} a_4 & a_0 \\ b_4 & b_0 \end{vmatrix} (4s^3) \end{aligned} \tag{36}$$

Equation (36) shows a special pattern of its terms. This pattern can be realized by setting up a template as shown in Figure 2.

i \ j			4	3	2	1
4			$(1)D_{4,3}s^6$	$(2)D_{4,2}s^5$	$(3)D_{4,1}s^4$	$(4)D_{4,0}s^3$
3				$(1)D_{3,2}s^4$	$(2)D_{3,1}s^3$	$(3)D_{3,0}s^2$
2					$(1)D_{2,1}s^2$	$(2)D_{2,0}s^1$
1						$(1)D_{1,0}s^0$

Figure 2. Template for fourth order system

Repeating the same steps for a fifth order system gives the four polynomials of 5-th order system and they are:

$$P_1(s) = -\sum_{q=1}^{n} q a_q s^{q-1} = -(a_1 + 2a_2 s + 3a_3 s^2 + 4a_4 s^3 + 5a_5 s^4) \quad (37)$$

$$P_2(s) = -\sum_{q=1}^{n} q b_q s^{q-1} = -(b_1 + 2b_2 s + 3b_3 s^2 + 4b_4 s^3 + 5b_5 s^4) \quad (38)$$

$$P_3(s) = a_0 - \sum_{q=1}^{n} (q-1) a_q s^q = a_0 - (a_2 s^2 + 2a_3 s^3 + 3a_4 s^4 + 4a_5 s^5) \quad (38)$$

$$P_4(s) = b_0 - \sum_{q=1}^{n} (q-1) b_q s^q = b_0 - (b_2 s^2 + 2b_3 s^3 + 3b_4 s^4 + 4b_5 s^5) \quad (40)$$

The substitution of the four polynomials, Equations (37-40), into Equation (8) gives the Break polynomial $P(s)$ such as

$$\begin{gathered} P(s) = (a_5b_4 - a_4b_5)(1s^8) + (a_5b_3 - a_3b_5)(2s^7) + (a_5b_2 - a_2b_5)(3s^6) + \\ +(a_5b_1 - a_1b_5)(4s^5) + (a_5b_0 - a_0b_5)(5s^4) + \\ +(a_4b_3 - a_3b_4)s^6 + 2(a_4b_2 - a_2b_4)s^5 + [3(a_4b_1 - a_1b_4) + (a_3b_2 - a_2b_3)]s^4 + \\ +[4(a_4b_0 - a_0b_4) + 2(a_3b_1 - a_1b_3)]s^3 + [3(a_3b_0 - a_0b_3) + (a_2b_1 - a_1b_2)]s^2 + \\ +2(a_2b_0 - a_0b_2)s + (a_1b_0 - a_0b_1) = 0. \end{gathered} \quad (41)$$

The Break polynomial, Equation (40), can be divided into five parts as follows:

$$\begin{aligned} &\text{1st:} && (a_5b_4 - a_4b_5)(1s^8) + (a_5b_3 - a_3b_5)(2s^7) + (a_5b_2 - a_2b_5)(3s^6) + +(a_5b_1 - a_1b_5)(4s^5) \\ & && + (a_5b_0 - a_0b_5)(5s^4) \\ &\text{2nd:} && (a_4b_3 - a_3b_4)(1s^6) + (a_4b_2 - a_2b_4)(2s^5) + (a_4b_1 - a_1b_4)(3s^4) + (a_4b_0 - a_0b_4)(4s^3) \\ &\text{3rd:} && (a_3b_2 - a_2b_3)(1s^4) + (a_3b_1 - a_1b_3)(2s^3) + (a_3b_0 - a_0b_3)(3s^2) \\ &\text{4th:} && (a_2b_1 - a_1b_2)(1s^2) + (a_2b_0 - a_0b_2)(2s) \\ &\text{5th:} && (a_1b_0 - a_0b_1)(1s^0) \end{aligned} \quad (42)$$

These five parts of Equation (42) may be written as follows:

$$\begin{aligned} &\text{1st:} && \begin{vmatrix} a_1 & a_0 \\ b_1 & b_0 \end{vmatrix}(1s^0) \\ &\text{2nd:} && \begin{vmatrix} a_2 & a_1 \\ b_2 & b_1 \end{vmatrix}(1s^2) + \begin{vmatrix} a_2 & a_0 \\ b_2 & b_0 \end{vmatrix}(2s^1) \\ &\text{3rd:} && \begin{vmatrix} a_3 & a_2 \\ b_3 & b_2 \end{vmatrix}(1s^4) + \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix}(2s^3) + \begin{vmatrix} a_3 & a_0 \\ b_3 & b_0 \end{vmatrix}(3s^2) \\ &\text{4th:} && \begin{vmatrix} a_4 & a_3 \\ b_4 & b_3 \end{vmatrix}(1s^6) + \begin{vmatrix} a_4 & a_2 \\ b_4 & b_2 \end{vmatrix}(2s^5) + \begin{vmatrix} a_4 & a_1 \\ b_4 & b_1 \end{vmatrix}(3s^4) + \begin{vmatrix} a_4 & a_0 \\ b_4 & b_0 \end{vmatrix}(4s^3) \\ &\text{5th:} && \begin{vmatrix} a_5 & a_4 \\ b_5 & b_4 \end{vmatrix}(1s^8) + \begin{vmatrix} a_5 & a_3 \\ b_5 & b_3 \end{vmatrix}(2s^7) + \begin{vmatrix} a_5 & a_2 \\ b_5 & b_2 \end{vmatrix}(3s^6) + \begin{vmatrix} a_5 & a_1 \\ b_5 & b_1 \end{vmatrix}(4s^5) + \begin{vmatrix} a_5 & a_0 \\ b_5 & b_0 \end{vmatrix}(5s^4) \end{aligned} \quad (43)$$

Based on Equation (43) the template is constructed as shown in Figure 3, and comparison of the template of the five-order system with the template of the four-order system shows that there is an addition of a top row in the template of the fifth order system. In a similar way the template for sixth order system is obtained by adding another top row to the template of the fifth order system as shown in Figure 4. There is resemblance between the mathematical expressions of the cells of the added top row and the mathematical expressions of the row under. The difference is that the determinant two indices are larger by one and is shifted to the left by one cell.

i \ j		5	4	3	2	1
5		$(1)D_{5,4}s^8$	$(2)D_{5,3}s^7$	$(3)D_{5,2}s^6$	$(4)D_{5,1}s^5$	$(5)D_{5,0}s^4$
4			$(1)D_{4,3}s^6$	$(2)D_{4,2}s^5$	$(3)D_{4,1}s^4$	$(4)D_{4,0}s^3$
3				$(1)D_{3,2}s^4$	$(2)D_{3,1}s^3$	$(3)D_{3,0}s^2$
2					$(1)D_{2,1}s^2$	$(2)D_{2,0}s^1$
1						$(1)D_{1,0}s^0$

Figure 3. Template for fifth order system

i \ j	6	5	4	3	2	1
6	$(1)D_{6,5}s^{10}$	$(2)D_{6,4}s^{9}$	$(3)D_{6,3}s^{8}$	$(4)D_{6,2}s^{7}$	$(5)D_{6,1}s^{6}$	$(6)D_{6,0}s^{5}$
5		$(1)D_{5,4}s^{8}$	$(2)D_{5,3}s^{7}$	$(3)D_{5,2}s^{6}$	$(4)D_{5,1}s^{5}$	$(5)D_{5,0}s^{4}$
4			$(1)D_{4,3}s^{6}$	$(2)D_{4,2}s^{5}$	$(3)D_{4,1}s^{4}$	$(4)D_{4,0}s^{3}$
3				$(1)D_{3,2}s^{4}$	$(2)D_{3,1}s^{3}$	$(3)D_{3,0}s^{2}$
2					$(1)D_{2,1}s^{2}$	$(2)D_{2,0}s^{1}$
1						$(1)D_{1,0}s^{0}$

Figure 4. Template for sixth order system

i/ j	6	5	4	3	2	1	
		s^{10}	s^{9}	s^{8}	s^{7}	s^{6}	s^{5}
6	$(1)D_{6,5}$	$(2)D_{6,4}$	$(3)D_{6,3}$	$(4)D_{6,2}$	$(5)D_{6,1}$	$(6)D_{6,0}$	s^{4}
5		$(1)D_{5,4}$	$(2)D_{5,3}$	$(3)D_{5,2}$	$(4)D_{5,1}$	$(5)D_{5,0}$	s^{3}
4			$(1)D_{4,3}$	$(2)D_{4,2}$	$(3)D_{4,1}$	$(4)D_{4,0}$	s^{2}
3				$(1)D_{3,2}$	$(2)D_{3,1}$	$(3)D_{3,0}$	s^{1}
2					$(1)D_{2,1}$	$(2)D_{2,0}$	s^{0}
1						$(1)D_{1,0}$	

Figure 5. Template for sixth order system where the variable factors are in the top row and the right column

To make it easier for the user and for the collection of the equal terms, the variable factors (s^l) are moved to the top row and to the right column of the template, while the cells are filled with the right determinant multiplied by its coefficient as shown in Figure 5. The parallel arrows are added to show the equal terms in each secondary diagonal. The sum of the cells of equal terms multiplied by the variable factor gives the terms of the Break polynomial.

It can be seen in the three figures (Figures 2 to 4), that there is a pattern of the terms in the cells of the template. An increase of the system order requires a filling of a new row on the top of the existing upper row of the same template. The first cell of the added row on the left column is filled by the expression $\{(1)D_{n,n-1}\, s^{2n-2}\}$ and the right cell is filled by the expression$\{(n)D_{n,0}\, s^{n-1}\}$. If the numbering of rows and columns in the template is flipped as shown in the previous three templates, the expression in each cell is

$$[(i-j+1)D_{i,j-1}s^{i+j-2}], \qquad \text{where } D_{i,j-1} = a_i b_{j-1} - b_i a_{j-1} \tag{44}$$

Then by collecting the equal terms we obtain the Break polynomial. This polynomial can be written as double summation of terms as follows

$$P(s) = \sum_{i=n}^{1} \sum_{j=i}^{1} [(i-j+1)D_{i,j-1}s^{i+j-2}] \tag{45}$$

This formula can be computer programmed if a right software with a symbolic toolbox is available, but in the absence of these tools another computer program can be written that is based on the Template method as shown in the flow chart, Figure 6. This computer program can be incorporated to the MATLAB root locus graphs which enhances the graph by presenting the exact values of the Break points on it.

3. TEMPLATE CONSTRUCTION

To calculate the determinants and fill in the cells at the same time, the template is expanded to include an array with two rows. The upper row is for the coefficients of the denominator's polynomial, and the lower row for the coefficients of the numerator polynomials. Also, a new row under is added for columns numbering, and a new column to the left for rows numbering. The numbering sequence is from right to left and from down up. The whole template is shown in Figure 7. The template can be organized and constructed in general as follows:

a. The number of column and rows are $n + 1$.
b. The upper row and the right column are for the variable factors (s^{n-1}).

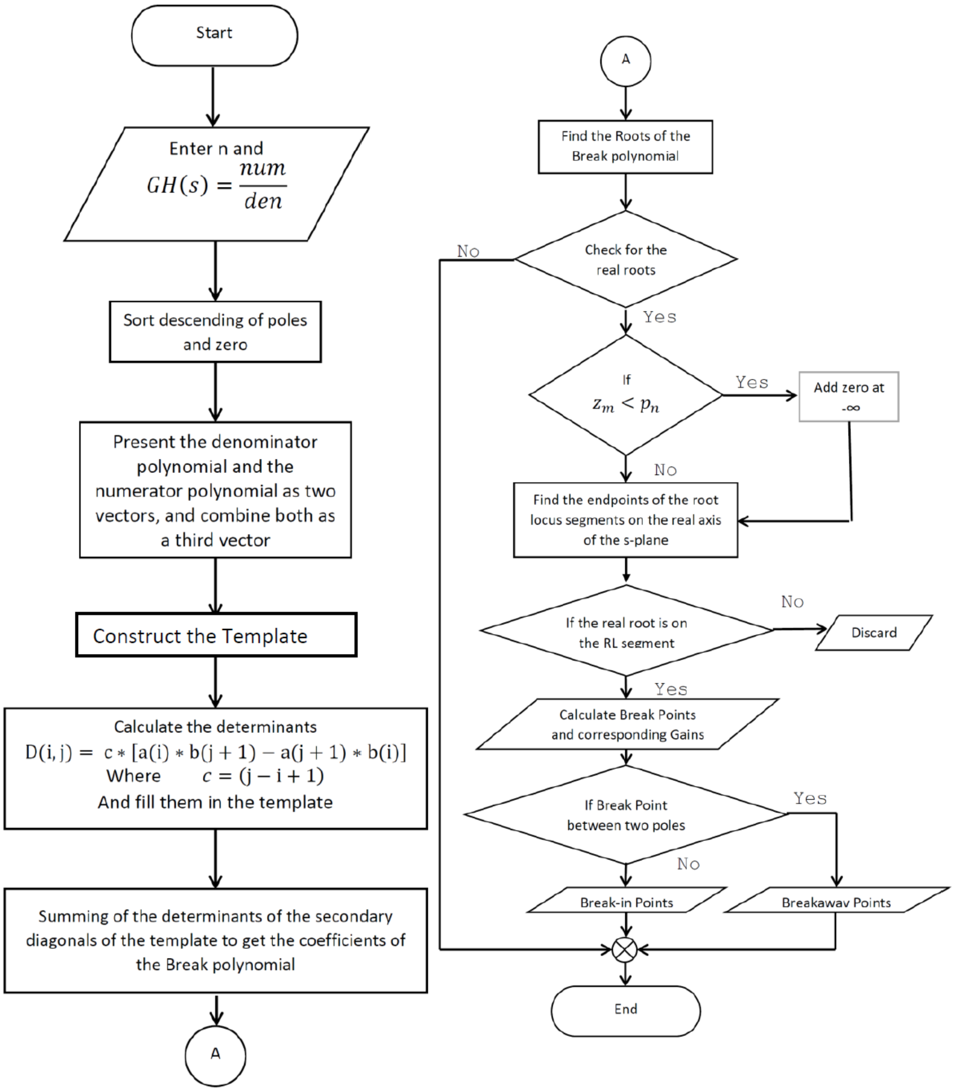


Figure 6. Computer flow chart to determine the Break polynomial, the Break points, and the corresponding gains

D	a_n	a_{n-1}	a_{n-2}	⋯	a_2	a_1	a_0
N	b_n	b_{n-1}	b_{n-2}	⋯	b_2	b_1	b_0
i \ j	n	n-1	n-2	⋯	2	1	0
		s^{2n-2}	s^{2n-3}	⋯	s^{n+1}	s^n	s^{n-1}
n	$(1)D_{n,n-1}$	$(2)D_{n,n-2}$	$(3)D_{n,n-3}$	⋯	$(n-1)D_{n,1}$	$(n)D_{n,0}$	s^{n-2}
n-1		$(1)D_{n-1,n-2}$	$(2)D_{n-1,n-3}$	⋯	$(n-2)D_{n-1,1}$	$(n-1)D_{n-1,0}$	s^{n-3}
n-2			$(1)D_{n-2,n-3}$	⋯	$(n-3)D_{n-2,1}$	$(n-2)D_{n-2,0}$	⋮
⋮					⋮	⋮	s^1
2					$(1)D_{0,1}$	$(2)D_{2,0}$	s^0
1						$(1)D_{1,0}$	

Figure 7. Template for sixth order system where the variable part of the terms is split

c. The variable factors are filled in the top row and the right column so that their common cell contains the variable factor.
d. The upper right half of the template is filled by the determinants and their coefficient as shown in inner frame of Figure 7.

The template looks complicated in general. To show the template simplicity of its construction and cells' filling for finding the Break polynomial, Break points, and the corresponding gains, is that by using the template method in the solution of the following two examples:

D		a_5	a_4	a_3	a_2	a_1	a_0
N		b_5	b_4	b_3	b_2	b_1	b_0
i\j		5	4	3	2	1	0
			s^8	s^7	s^6	s^5	s^4
5		$(1)D_{5,4}$	$(2)D_{5,3}$	$(3)D_{5,2}$	$(4)D_{5,1}$	$(5)D_{5,0}$	s^3
4			$(1)D_{4,3}$	$(2)D_{4,2}$	$(3)D_{4,1}$	$(4)D_{4,0}$	s^2
3				$(1)D_{3,2}$	$(2)D_{3,1}$	$(3)D_{3,0}$	s^1
2					$(1)D_{2,1}$	$(2)D_{2,0}$	s^0
1						$(1)D_{1,0}$	

(a)

D		1	12	49	78	40	0
N		0	0	1	17	90	144
i/j		5	4	3	2	1	0
			s^8	s^7	s^6	s^5	s^4
5		0	2	51	360	720	s^3
4			12	408	3240	6912	s^2
3				755	8740	21168	s^1
2					6340	22464	s^0
1						5760	

(b)

Figure 8. Template for Example 1, (a) in general, (b) specific for the example

4. SOLUTION OF EXAMPLES IN THREE METHODS

To show the simplicity, and the amount of work the Template method is compared with other common methods in solving of two examples. The solution of the two examples is done by three methods; the Template method, MATLAB plotting of root locus graph, and by Franklin via Transition method.

4.1 Solution of Example 1

4.1.1 Using the Template Method

Considering the following fifth order control system

$$G(s)H(s) = K\frac{(s+3)(s+6)(s+8)}{s(s+1)(s+2)(s+4)(s+5)} = \frac{s^3+17s^2+90s+144}{s^5+12s^4+49s^3+78s^2+40s} \quad (46)$$

The general form of the Template of this fifth order control system is shown in Figure 8(a). The cells are filled by the determinants multiplied by their coefficients. The arrows are drawn for beginner to show the secondary diagonals for adding their elements together. Then after several practice there in no need for having the arrows. The Template of this example is shown in Figure 8(b). The sum of the terms of each secondary diagonal as shown by the arrows are the coefficient of the corresponding variable factor, and then adding the results to get the Break polynomial. After 45 multiplications and 21 additions the Break polynomial is obtained such as

$$P(s) = 2s^7 + 63s^6 + 768s^5 + 4715s^4 + 15652s^3 + 27508s^2 + 22464s + 5760 \quad (47)$$

The roots of the Break polynomial are: [-0.4456, -1.6192, -3.2991 - 0.6479i, -3.2991 + 0.6479i, -4.6480, -6.4986, -11.6904]. The segments of the root locus graph that satisfy the angle condition on the real axis in this example are: $[0 \quad -1], [-2 \quad -3], [-4 \quad -5], [-6 \quad -9]$. The segment that may contains a Break point is the segment that its two ends are poles or zeros.

In this example those segments are $[0 \quad -1]$, $[-4 \quad -5]$, $[-6 \quad -9]$. The real roots that are on these segments are: (-0.4456, -4.6480, -6.4986). Then solving *K* in Equation (5) for those points to obtain the corresponding gain as [0.057997, 1.371256, and 229.820517]. A Breakaway point is located on a segment which its two ends are poles such as ((-0.4456, -4.6480), and a Break-in point is located on a segment which its two ends are zeros such as (-6.4986).

4.1.2 Using the MATLAB Software

Another graphical solution of this example is plotting root locus graph using MATLAB. The graph is shown in Figure 9.

4.1.3 Using Franklin via Transition Method

The formula used in this method is

$$\sum_{i=1}^{n}\frac{1}{s+p_i} = \sum_{i=1}^{m}\frac{1}{s+z_i} \quad (48)$$

Applying this formula to this example gives

$$\frac{1}{s}+\frac{1}{s+1}+\frac{1}{s+2}+\frac{1}{s+4}+\frac{1}{s+5} = \frac{1}{s+3}+\frac{1}{s+6}+\frac{1}{s+8} \quad 49)$$

Fraction addition gives

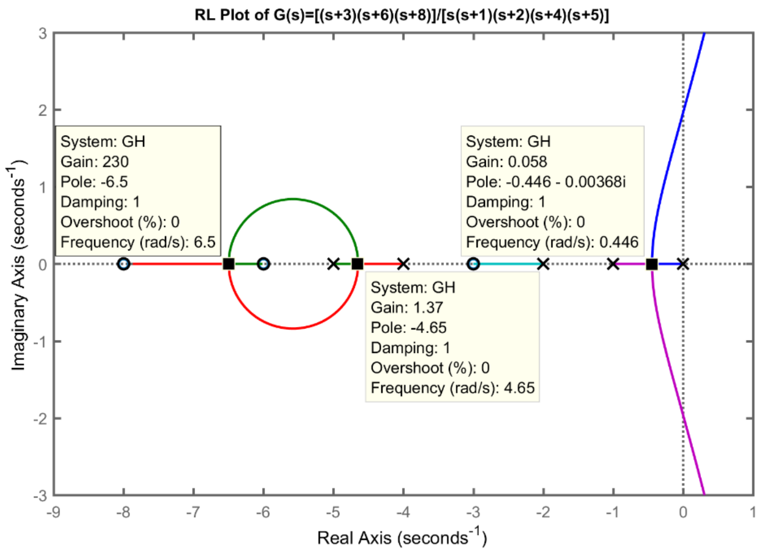


Figure 9. Root Locus Plot of example 1 showing the Break points and the corresponding gains

$$\frac{(s+1)(s+2)(s+4)(s+5)+s(s+2)(s+4)(s+5)+s(s+1)(s+4)(s+5)+s(s+1)(s+2)(s+5)+s(s+1)(s+2)(s+4)}{s(s+1)(s+2)(s+4)(s+5)} = \frac{(s+6)(s+8)+(s+3)(s+8)+(s+3)(s+6)}{(s+3)(s+6)(s+8)} \quad (50)$$

Cross multiplication gives

$$[(s+1)(s+2)(s+4)(s+5)+s(s+2)(s+4)(s+5)+s(s+1)(s+4)(s+5)+s(s+1)(s+2)(s+5)+ +s(s+1)(s+2)(s+4)][(s+3)(s+6)(s+8)] - [s(s+1)(s+2)(s+4)(s+5)][(s+6)(s+8)+(s+3)(s+8)+(s+3)(s+6)] = 0 \quad (51)$$

After 160 mathematical multiplications and 100 additions the Break polynomial is obtained as

$$P_F(s) = 2s^7 + 63s^6 + 768s^5 + 4715s^4 + 15652s^3 + 27508s^2 + 22464s + 5760 = 0 \quad (52)$$

It is same polynomial as the polynomial obtained by the Template method.

4.2 Solution of Example 2

To show the Template modification for a higher order system, a system of order six was chosen. The open loop transfer function of the chosen system has pair of complex conjugate poles. The intention of transfer function with complex poles is to show that the template method can handle also complex roots. In this example the transfer function is

$$G(s)H(s) = K\frac{(s+3)(s+4)}{s(s+1+j)(s+1-j)(s+2)(s+5)(s+6)} = \frac{s^3+7s^2+12}{s^6+15s^5+80s^4+190s^3+224s^2+120s} \quad (53)$$

4.2.1 Using Template Method

The template for this sixth order system of this example is shown in Figure (10). Adding the secondary diagonals elements to get the Break polynomial as

$$P(s) = 4s^7 + 80s^6 + 652s^5 + 2770s^4 + 6500s^3 + 8288s^2 + 5376s + 1440 \quad (54)$$

The roots of the polynomial are:[-0.7576 - 0.3625i, -0.7576 + 0.3625i, -1.5082, -3.4166, -4.0415 - 1.3223i, -4.0415 + 1.3223i, -5.4769]. The real roots are [-1.5082, -3.4166, -5.4769]. The Breakaway points are: [-1.508234, -5.476910] and the Break-in point is [-3.416561]. The corresponding gains are: [K=3.937915, 27.326221, and 557.174532].

4.2.2 Using MATLAB Software|

The root locus plot of the given transfer function of this example is shown in Figures 11(a) and 11(b).

4.2.3 Using Franklin via Transition Method

Applying the formula to this example gives

$$\frac{1}{s}+\frac{1}{s+1+j}+\frac{1}{s+1-j}+\frac{1}{s+2}+\frac{1}{s+5}+\frac{1}{s+6}=\frac{1}{s+3}+\frac{1}{s+4} \tag{55}$$

or

$$\frac{1}{s}+\frac{2s+2}{s^2+2s+2}+\frac{1}{s+2}+\frac{1}{s+5}+\frac{1}{s+6}=\frac{1}{s+3}+\frac{1}{s+4} \tag{56}$$

Algebraic simplification gives

$$\frac{(s^2+2s+2)(s+2)(s+5)(s+6)+s(2s+2)(s+2)(s+5)(s+6)+s(s^2+2s+2)(s+5)(s+6)+}{s(s^2+2s+2(s+2)(s+5)(s+6)}+ \\ +\frac{s(s^2+2s+2)(s+2)(s+6)+s(s^2+2s+2)(s+2)(s+5)}{s(s^2+2s+2(s+2)(s+5)(s+6)}=\frac{(s+4)+(s+3)}{(s+3)(s+4)} \tag{57}$$

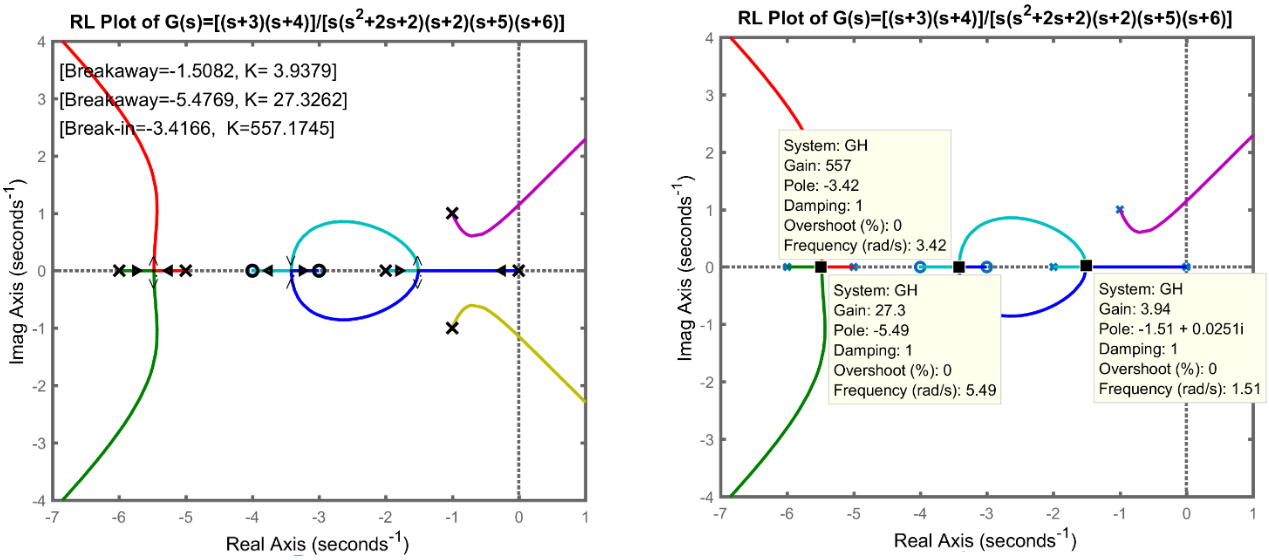
Cross multiplication gives

$$[(s^2+2s+2)(s+2)(s+5)(s+6)+s(2s+2)(s+2)(s+5)(s+6)+s(s^2+2s+2)(s+5)(s+6)+ \\ s(s^2+2s+2)(s+2)(s+6)+s(s^2+2s+2)(s+2)(s+5)][(s+3)(s+4)]-[s(s^2+2s+2(s+2)(s+ \\ 5)(s+6)][(s+4)+(s+3)]=0 \tag{58}$$

D	1	15	80	190	224	120	0
N	0	0	0	0	1	7	12

i/ j	6	5	4	3	2	1	0
		s^{10}	s^9	s^8	s^7	s^6	s^5
6	0	0	0	4	35	72	s^4
5		0	0	45	420	900	s^3
4			0	160	1680	3840	s^2
3				190	2660	6840	s^1
2					1448	5376	s^0
1						1440	

Figure 10. Template of Example 2



(a) (b)

Figure 11. Root Locus plot of example 2 showing the Break points and the corresponding gains

or

$$P_F(s) = 4s^7 + 80s^6 + 652s^5 + 2770s^4 + 6500s^3 + 8288s^2 + 5376s + 1440 = 0 \quad (59)$$

It is the same polynomial as the polynomial obtained by the Template method. To obtain this polynomial by Via-Transition method requires 348 mathematical multiplications and 174 additions, while in the Template method it requires 63 multiplications and 31 additions, and if the zero multiplication is not counted then in this example the number of multiplication and addition is the same as in Example 1.

5. CONCLUSION

The proof of the correctness of the bases of the Template method used for any order of the control system is presented. As a result, the Break polynomial, Break points, and the corresponding gains obtained by using the Template method also are correct. Two examples were solved to demonstrate the method's technique, the simplicity, and the correct results that obtained by the proposed method. To validate the results of the two examples they are solved also using the root locus graphs that plotted by MATLAB, and via Transition method of Franklin. The root locus plots show that the values of the Break points are too close to the result obtained from the Break polynomial of the Template method. The small differences are a result of the root locus method being a semi-graphical method. The results of the two examples' solution by the Template method and by Franklin via Transition methods are the same. While the number of mathematical operations as multiplication, and addition to obtain the result requires 28% for multiplications and 21% for addition in the Template method compared to the mathematical operations that is required by the Via Transition method for an $n = 6$. Increasing the system order decrease the percentage to about 18% for both types of mathematical operations. The number of mathematical operations is negligible for today computers, but it is significant for students and engineers hand solution of problems and design, because, for many mathematical operations the work becomes tedious and the probability of making errors is higher. In addition, the Template method can handle also complex poles or zeros as shown in the second example where the system has two complex poles. In addition to that it was shown two approaches for using the Template method as computerized and for hand calculation.

To show the systematic modification of the Template when increasing the system order two systems were chosen one is fifth order system and second is sixth order system. The modification to obtain the template for a system of higher order of one than the existing template is obtained by adding a top row to the existing template. The mathematical expressions of the cells of the added top row are the mathematical expressions of the row under with higher indices by one and by shifting the cells to the left by one cell.

It can be concluded that the Template method is a simple technique and is a systematic method for finding all Break points, breakaway, and break-in points. In the Template method there is no mathematical differentiation, and for a constant numerator all the coefficients are zeros which simplifies drastically the determinants calculation. After several use of this method the technique becomes easier for obtaining the Break polynomial. In addition, the template method can be computer programmed and if it is incorporated to the MATLAB root locus graphs it enhances the graph by presenting the exact values of the Break points on it. This method is another accurate method which gives the exact values of all Break points. It has its own merits which are different from the commonly used methods, such as its simplicity by filling a table without performing calculus operations. This method will be favorable for users not comfortable with calculus. It can be computer programed and incorporated in root locus plots of MATLAB software.

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APPENDIX

The correctness of the bases of the Template method is proved by showing that the Template method's Break polynomial is the same polynomial that achieved by other method. This means that $P(s)$, the Break polynomial of the Template method, is the same polynomial $P_F(s)$ of the Via Transition method. This is achieved by showing that both sides of the following equation are the same.

$$P(s) \equiv P_F(s) \quad (A1)$$

Where, the symbol ≡ is for questioning the equality of the two sides of the equation which must be proved next. Substitute both polynomials, Equation (19) and Equation (28), into Equation (A1) to have

$$-\left(\sum_{q=1}^{n} q a_q s^{q-1}\right)\left(b_0 - \sum_{q=1}^{n}(q-1) b_q s^q\right) + \left(\sum_{q=1}^{n} q b_q s^{q-1}\right)\left(a_0 - \sum_{q=1}^{n}(q-1) a_q s^q\right) \overset{?}{\triangleq}$$

$$\left(\sum_{q=1}^{m} q b_q s^{q-1}\right)\left(a_0 + \sum_{q=1}^{n} a_q s^q\right) - \left(\sum_{q=1}^{n} q a_q s^{q-1}\right)\left(b_0 + \sum_{q=1}^{m} b_q s^q\right) \quad \text{(A2)}$$

Algebraic simplifications of both sides of Equation (A2) gives

$$-b_0\left(\sum_{q=1}^{n} q a_q s^{q-1}\right) + \left(\sum_{q=1}^{n} q a_q s^{q-1}\right)\left(\sum_{q=1}^{n}(q-1) b_q s^q\right) + a_0\left(\sum_{q=1}^{n} q b_q s^{q-1}\right) -,$$

$$\left(\sum_{q=1}^{n} q b_q s^{q-1}\right)\left(\sum_{q=1}^{n}(q-1) a_q s^q\right) \overset{?}{\triangleq} a_0\left(\sum_{q=1}^{m} q b_q s^{q-1}\right) + \left(\sum_{q=1}^{m} q b_q s^{q-1}\right)\left(\sum_{q=1}^{n} a_q s^q\right) -,$$

$$b_0\left(\sum_{q=1}^{n} q a_q s^{q-1}\right) - \left(\sum_{q=1}^{n} q a_q s^{q-1}\right)\left(\sum_{q=1}^{m} b_q s^q\right) \quad \text{(A3)}$$

The removing of equal terms from both sides of Equation (A3) gives

$$\left(\sum_{q=1}^{n} q a_q s^{q-1}\right)\left(\sum_{q=1}^{n}(q-1) b_q s^q\right) - \left(\sum_{q=1}^{n} q b_q s^{q-1}\right)\left(\sum_{q=1}^{n}(q-1) a_q s^q\right) \overset{?}{\triangleq},$$

$$\left(\sum_{q=1}^{m} q b_q s^{q-1}\right)\left(\sum_{q=1}^{n} a_q s^q\right) - \left(\sum_{q=1}^{n} q a_q s^{q-1}\right)\left(\sum_{q=1}^{m} b_q s^q\right) \quad \text{(A4)}$$

Split each of the following two expressions of Equation (A4) to two terms of summation as follows:

$$\left(\sum_{q=1}^{m}(q-1) b_q s^q\right) = \left(\sum_{q=1}^{m} \mathrm{q} b_q s^q\right) - \left(\sum_{q=1}^{m} b_q s^q\right) \quad \text{(A5)}$$

$$\left(\sum_{q=1}^{n}(q-1) a_q s^q\right) = \left(\sum_{q=1}^{n} \mathrm{q} a_q s^q\right) - \left(\sum_{q=1}^{n} a_q s^q\right) \quad \text{(A6)}$$

Substitute Equations (A5-A7) into Equation (A4) to obtain

$$\left(\sum_{q=1}^{n} q a_q s^{q-1}\right)\left(\sum_{q=1}^{m} q b_q s^q\right) - \left(\sum_{q=1}^{n} q a_q s^{q-1}\right)\left(\sum_{q=1}^{m} b_q s^q\right) - \left(\sum_{q=1}^{m} q b_q s^{q-1}\right)\left(\sum_{q=1}^{n} q a_q s^q\right) +,$$

$$+\left(\sum_{q=1}^{m} q b_q s^{q-1}\right)\left(\sum_{q=1}^{n} a_q s^q\right) \overset{?}{\triangleq} \left(\sum_{q=1}^{m} q b_q s^{q-1}\right)\left(\sum_{q=1}^{n} a_q s^q\right) - \left(\sum_{q=1}^{n} q a_q s^{q-1}\right)\left(\sum_{q=1}^{m} b_q s^q\right) \quad \text{(A7)}$$

By removing equal terms in both sides of Equation (A7) gives

$$\left(\sum_{q=1}^{n} q a_q s^{q-1}\right)\left(\sum_{q=1}^{m} q b_q s^q\right) - \left(\sum_{q=1}^{m} q b_q s^{q-1}\right)\left(\sum_{q=1}^{n} q a_q s^q\right) \overset{?}{\triangleq} 0 \quad \text{(A8)}$$

By taking out "s" as a factor in Equation (A8) gives

$$s\left[\left(\sum_{q=1}^{n} q a_q s^{q-1}\right)\left(\sum_{q=1}^{m} q b_q s^{q-1}\right) - \left(\sum_{q=1}^{m} q b_q s^{q-1}\right)\left(\sum_{q=1}^{n} q a_q s^{q-1}\right)\right] \overset{?}{\triangleq} 0 \quad \text{(A9)}$$

The two expressions of the two products within the brackets in Equation (A9) are the same. So, their subtraction gives zero value. Then Equation (A9) becomes

$$(0)s = 0 \quad \text{(A10)}$$

The result of Equation (A10) is a mathematical proof which shows that the Template method gives the same Break polynomial as the Break polynomial which is obtained by the popular Via Transition method. Therefore, the proposed Template method is correct and accurate method.