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Polynomial Estimation of J-integral for Through-thickness Crack in Elastic Perfectly-Plastic Conditions

Norwahida Yusoff* and Feizal Yusof

School of Mechanical Engineering, Universiti Sains Malaysia, 14300 Nibong Tebal, Penang, Malaysia

*Corresponding author: menorwahida@usm.my

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Abstract: Through the means of finite element analysis, J-integral for a straight through-thickness crack has been consistently reported to decrease along the crack front with the maximum and minimum values are at the midplane and the free-surface, respectively. The present study aims to examine the through-thickness profile of J-integral and represent it mathematically that it could be used to replace the demanding works of finite element analysis. The J-integral profile was numerically examined using finite element analysis of three-dimensional boundary layer formulation that is subjected to a uniform load on the outermost boundary of the model. The results verify that in three-dimensional cracked bodies, although the applied load is uniform, the intensity of deformation in terms of J-integral varies along the crack front and the profile has been expressed in a polynomial equation. The analytical solution allows experimentalists to estimate the crack-tip deformation in terms of local J for a known value of J that is applied on a related test specimen.

Keywords: Finite element; J-integral; Polynomial; Three-dimensional; Through-thickness crack.

1. INTRODUCTION

J-integral is a crack-tip parameter proposed by Rice [1] to quantify the severity of crack-tip strain and stress fields in elastic-plastic materials. As originally formulated for two-dimensional, the parameter is defined as a path independent line integral describing the amount of energy released during the development of a crack that is evaluated along an arbitrary contour around the crack-tip. Determination of J-integral in cracked bodies demands a meticulous analytical computation that is mostly in conjunction with rigorous finite element analyses. The J-integral estimation schemes based on the elastic-plastic finite element analyses for a wide range of specimen geometries in two-dimensional plane stress and plane strain conditions have been documented in [2]. Also, through the finite element method, Parks [3] proposed an alternative technique referred to as virtual crack extension (VCE) to determine the J-integral based on an energy comparison between an altered crack length, which was modified from the technique of measuring the stress intensity factor in an elastic crack-tip [4, 5]. Detailed review on the development of the VCE technique can be found in [6].

To consider the out-of-plane stress in three-dimensional cases, the VCE technique was extended to three-dimensional finite element model by computing J-integral as an equivalent domain integral (EDI) converted from a line integral using a divergence theorem [7, 8]. The resulting integral was in congruent with that calculated by DeLorenzi [9]. It was observed that the J-integral in two-dimensional and three-dimensional cracks becomes path dependent particularly under incremental plasticity [10, 11]. In that sense, the local J-integral at a given position along the crack front will increase as the distance from the crack tip increases until a saturated value is approached. The saturated value of J-integral is close to the applied J-integral at far field. The determination of J along a three-dimensional crack front using the EDI method has been widely implemented in most finite element packages and used in numerous studies for various crack conditions. Examples of the determination of J-integral beyond a straight through-thickness crack can be seen from studies involving curved crack fronts [12], non-planar cracks under mixed mode [13] or interface crack in a bimaterial plate [14].

For a straight through-thickness crack, Nakamura and Parks [15] reported that, under different load levels in non-hardening conditions, the normalized J-integral attained its maximum value at the midplane of a specimen and gradually decreased along the crack front with its minimum value at the free-surface. The same observation of J-integral variation along the crack front of the same crack condition have been reported in [16-19]. This consistency implies that the variation of J-integral as a function of loads in a straight through-thickness crack is unique that it can be expressed mathematically. This would eliminate the need of numerical finite element calculation in determining J-integral which is often cumbersome and a very demanding task. Therefore, the present study is aimed to formulate an analytical estimation of J-integral specifically for the most common straight through-thickness crack under an elastic perfectly-plastic condition. The idealized perfectly-plastic solution is preferred in this study as it limits the crack-tip deformation fields to the minimum bound that a material can develop to give a conservative estimate, which is crucial in structural design. The mathematical formulation will allow experimentalists to estimate the intensity of deformation fields in terms of J-integral that is generated at the crack-tip of a related test specimen under an applied load.

2. FINITE ELEMENT ANALYSIS

Boundary layer formulation (BLF) is a method of analyzing crack-tip field that was proposed by Rice [20, 21] to approximate a crack-tip domain under a small scale yielding condition. Therefore, in finite element analysis, the method of boundary layer formulation models the crack-tip region as an annular region without having to model the whole cracked body, and the dominant elastic stress field is generated by applying a displacement corresponding to the elastic field on the outermost traction as a boundary condition.

In this study, the boundary layer formulation method was used to examine the through-thickness profile of J-integral in a thin plate that is subjected to loadings in a non-hardening small-scale yielding condition. The near tip region of a thin plate was modeled with a circular disk of thickness t, as illustrated in Figure 1. The disk is centered at the crack-tip at the midplane of the plate (x_3/t = 0), where the straight crack front lies along the x_3-axis. The maximum radial extent r of the disk was considered so that the plastic zone remains well-contained to a maximum of 10% of the radial length of the disk. As the problem has reflective symmetry with respect to the midplane and the crack plane (x_2 = 0), only a quarter of the circular disk (region $0 \leq \theta \leq \pi$, $0 \leq x_3/t \leq 0.5$) was modeled with finite elements.

The three-dimensional finite element model of sharp crack-tip BLF is shown in Figure 2, which is relatively similar to that employed by Nakamura and Parks [15] with some model improvement. The three-dimensional mesh was built from a 20-node hexahedron isoparametric quadratic brick element, with reduced integration and linear pressure interpolation (element type C3D20RH) provided in the finite element code ABAQUS [22]. In order to attain a highly refined crack-tip mesh with optimum element aspect ratio, three sub-structured finite element meshes, referred as outer-mesh, middle-mesh, and tip-mesh, were created using submodeling technique as implemented in ABAQUS [22]. The outer-mesh was one element thick, and further sub-modeled to the middle-mesh, which was disposed in ten element layers of uniform thickness. The tip-mesh, which encompassed the crack-tip, was constructed with high density of mesh near the crack-tip to capture sensitive deformation accurate solutions of field variables near the crack front. The tip-mesh was arranged in 32 gradually decreasing layers in x_3-direction, giving the thickness of the free-surface element of $0.5t/1000$. The crack-tip was modeled as a sharp crack-tip through a focused mesh with 25 coincident but independent crack-tip nodes, as shown in Figure 2. In total, the whole model was constructed with 11664 20-noded brick elements.

A dominant elastic stress field surrounding the crack-tip plasticity is generated by applying in-plane displacements u_1 and u_2 of elastic plane stress K_I-field (Equation (1)) on the nodes of the outer perimeter of the outer-mesh as illustrated in Figure 2. The out-of-plane displacement, u_3, was allowed to remain a free variable. Due to the symmetry conditions, the displacements normal to the crack plane were restrained at all nodes on the crack plane including the crack-tip node. The submodeling analysis card allows the loading applied to the coarsest outer-mesh to be used to drive the subsequent more refined meshes. The displacements from the outer-mesh were interpolated to the nodes on the outer boundary of the middle-mesh. Subsequently, the displacements interpolated from the middle-mesh were used to drive the nodes on the tip-mesh boundary.

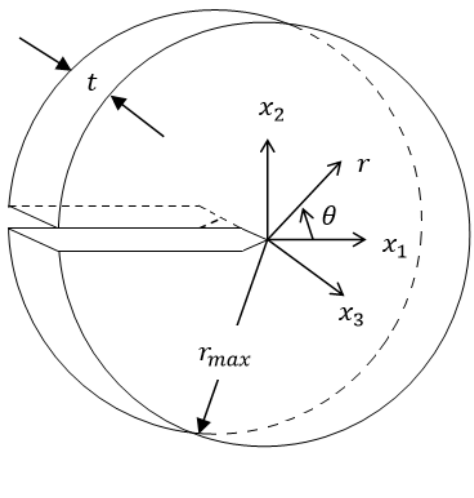

Figure 1. The circular disk representing the near crack front region of a thin plate

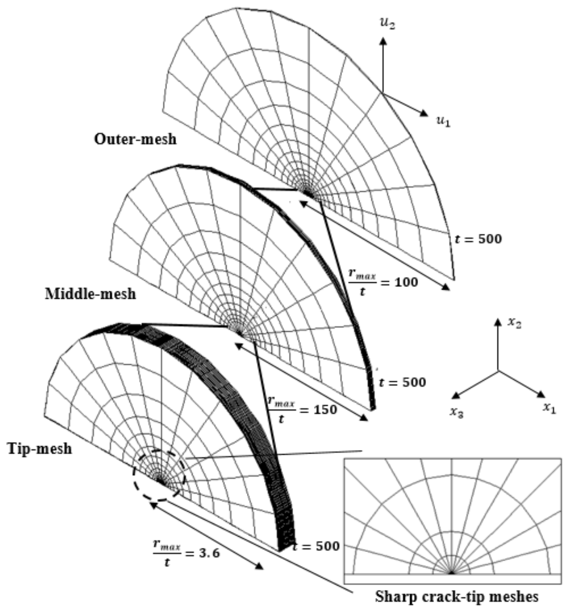

Figure 2. The finite element meshes of boundary layer formulation with a sharp crack-tip

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \frac{K_I}{2G}\sqrt{\frac{r}{2\pi}} \begin{Bmatrix} cos\left(\frac{\theta}{2}\right)\left[\kappa - 1 + 2sin^2\left(\frac{\theta}{2}\right)\right] \\ sin\left(\frac{\theta}{2}\right)\left[\kappa + 1 - 2sin^2\left(\frac{\theta}{2}\right)\right] \end{Bmatrix} \tag{1}$$

The J-integral at the outer boundary, J_{far}, was related to K_I by the relationship:

$$J_{far} = \frac{K_1^2}{E} \tag{2}$$

The deformation level is quantified by non-dimensional loading parameter, Ω_{far}, of the form:

$$\Omega_{far} = \frac{J_{far}}{\sigma_o \varepsilon_o t} \tag{3}$$

where σ_o is the yield stress, ε_o is the yield strain, and t is the plate thickness. The analyses were carried out at four level of deformations, Ω_{far} = 1, 3, 5, and 8. The values of local J, J_{loc}, along the crack front were determined by domain integral methods implemented in ABAQUS [22] using 26 contours, which was sufficient to ensure path independence.

For the three-dimensional boundary layer formulations, the material response was idealized as elastic perfectly-plastic based on uni-axial idealization of the form:

$$\sigma = E\varepsilon \quad \sigma \leq \sigma_o \quad ; \qquad \sigma = \sigma_o \quad \sigma \geq \sigma_o \tag{4}$$

Here σ is the uni-axial stress, σ_o is the yield stress, ε is strain and E is Young's modulus. The uniaxial stress-strain relations were generalized for multi-axial stress state using the Mises yield criterion with an associated flow rule. Therefore, in this analysis, the material was assumed homogenous isotropic and nearly incompressible with a Poisson's ratio of 0.49 in order to reduce volumetric locking that would occur in full integrated finite elements when ν = 0.5. The analysis was carried out with Young's Modulus, E = 200 GPa, and yield stress, σ_o = 200 MPa, yet most results were presented in non-dimensional forms to permit general applicability of the results.

3. MATHEMATICAL FORMULATION

In small scale yielding of an elastic perfectly-plastic crack-tip deformation, the elastic field is driven by the stress intensity factor, K which dominates the far field. Although the remote deformation is uniform, the intensity of deformation varies along the crack front, which is quantified by the local values of J-integral, J_{loc}. The non-dimensional deformation at a local level is interpreted as Ω_{far} when J is treated as a local J, following Equation (3). Figure 3 shows the variations of normalized J_{loc} along the crack front at all deformation levels, Ω_{far} = 1, 3, 5, 8, which appear to increase over the remotely applied J_{far} near the midplane but attenuated near the free-surface. Near the quarter-plane, the ratio of J_{loc}/J_{far} at all levels of deformation is approximately unity and decreases as the free-surface is approached.

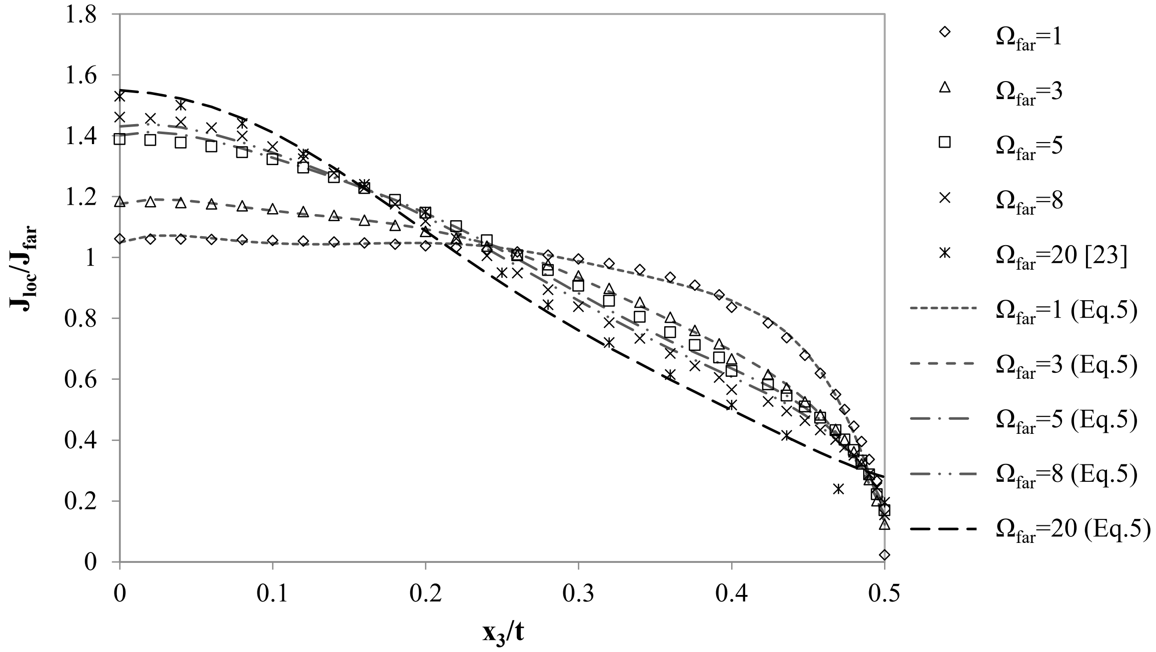

Figure 3. The variation of normalized local J along the crack front as a function of load levels

The normalized local J, J_{loc}/J_{far}, along the crack front as a function of applied load, Ω_{far}, may be expressed mathematically using curve-fitting method. All curves of J_{loc}/J_{far} vs. x_3/t are fitted with a six-order polynomial equation that gives seven different constants for each Ω_{far}. The relation between these constants with respect to Ω_{far} is determined separately for two groups of applied load: (i) small load, $\Omega_{far} < 5$, and (ii) medium to large load, $5 \leq \Omega_{far} \leq 20$. The constants for both groups appear to be linearly related to Ω_{far}. The constant- Ω_{far} linear relations are then substituted into the polynomial equation that gives:

$$\frac{J_{loc}}{J_{far}} = \left[\left(A\Omega_{far} + a\right)\left(\frac{x_3}{t}\right)^6\right] + \left[\left(B\Omega_{far} + b\right)\left(\frac{x_3}{t}\right)^5\right] + \left[\left(C\Omega_{far} + c\right)\left(\frac{x_3}{t}\right)^4\right] + \left[\left(D\Omega_{far} + d\right)\left(\frac{x_3}{t}\right)^3\right] + \left[\left(E\Omega_{far} + e\right)\left(\frac{x_3}{t}\right)^2\right] + \left[\left(F\Omega_{far} + f\right)\left(\frac{x_3}{t}\right)\right] + \left[G\Omega_{far} + g\right] \tag{5}$$

where the constants A, a, B, b, C, c, D, d, E, e, F, f, G, and g are dependent on the load level, and their values are given in Table 1.

Table 1. Values of the constants in Equation (5)

Constants	$\Omega_{far} < 5$	$5 \leq \Omega_{far} \leq 20$
A	437.15	176.24
a	-3401.3	-2826.5
B	-545.55	-241.43
b	4496.5	3803
C	265.95	121.74
c	-2264.4	-1911.7
D	-62.055	-26.824
d	528.84	446.74
E	5.7475	2.3463
e	-55.693	-52.673
F	-0.2658	-0.1096
f	2.1903	1.8481
G	0.0633	0.0099
g	0.9865	1.3517

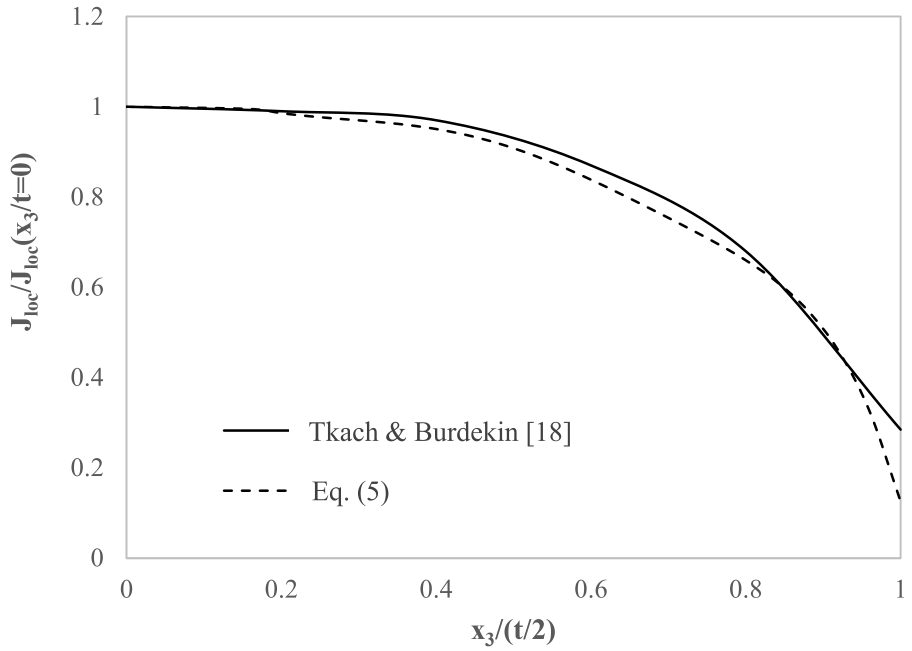

Figure 4. Through-thickness variation of normalized J_{loc} for J_{far} = 100 N/mm in a 25 mm CT specimen, estimated using Equation (5), and presented by Tkach and Burdekin [18]

For comparison, the local deformation for each load level estimated using Equation (5) is plotted in the dashed lines as shown in Figure 3. Slight deviations from the numerical values, which are less than 3 percent, are only noticeable near the midplane for load levels Ω_{far} = 5, and 8. For lower load levels Ω_{far} = 1, and 3, both solutions predict relatively similar level of local deformation across the thickness. Additionally, at a very large load level Ω_{far} = 20, the through-thickness variation of local J estimated using the polynomial equation appears to be comparatively parallel with the deformation field reported by Yusof [23] that was obtained through finite element analysis.

4. VERIFICATION OF MATHEMATICAL SOLUTION

To demonstrate the applicability of the formulated mathematical solution (Equation (5)) clearly, data from the work of Tkach and Burdekin [18] is taken to assessment. Since the material used in the study is a low hardening (n = 15), this reassessment is mostly intended to show the practicability of the solutions and expecting only proximity results. One of the applied J used in the study, J_{far} = 10 N/mm, is considered that corresponds to deformation levels of Ω_{far}= 2.34, and $J_{ave}/b\sigma_o$ = 0.014. Using Equation (5), the values of local J, J_{loc}, along the crack front of a 25 mm thickness CT specimen with σ_o = 585 MPa, and E = 200 GPa is analytically estimated. Figure 4 shows the estimated J_{loc} along the crack front that is normalized with J_{loc} at the midplane (x_3/t = 0), and compared with that of Tkach and Burdekin [18] at the deformation level $J_{ave}/b\sigma_o$ = 0.0149. The results show that at the same level of deformation, the variation of the analytically-estimated J_{loc} in most sections along the crack front of the CT specimen is comparatively similar to that of Tkach and Burdekin [18], which is about 3 percent accurate, although the estimated value at the free-surface of the specimen is noticeably lower. This, therefore, validates the credibility of Equation (5) for estimation of crack-tip deformation along the crack front of a test specimen in terms of J-integral.

5. CONCLUSION

In a three-dimensional cracked body that is subjected to small scale yielding of an elastic perfectly-plastic condition, the local crack-tip deformation in terms of J-integral varies along the crack front although the applied load is constant. A polynomial estimation of the through-thickness variation of J-integral has been credibly formulated that will optimistically allow experimentalists to estimate the crack-tip J-integral for a known value of J that is applied on a related test specimen. Therefore, with a proper implementation, the analytical estimation of three-dimensional crack-tip deformation as expressed by the local J-integral can be a favorable tool in fracture mechanics.

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REFERENCES

[1] J. R. Rice, A path independent integral and the approximate analysis of strain concentration by notches and cracks, *Journal of Applied Mechanics,* 35(2), 1968, 379-386.

[2] V. Kumar, M. D. German and C. F. Shih, *Engineering Approach For Elastic-Plastic Fracture Analysis,* Technical report EPRI-NP-1931, General Electric Co., Schenectady, NY, USA, 1981.

[3] D. M. Parks, The virtual crack extension method for nonlinear material behavior, *Computer Methods in Applied Mechanics and Engineering,* 12, 1977, 353-364.

[4] D. M. Parks, A toughness derivative finite element technique for determination of crack tip stress intensity factors, *International Journal of Fracture,* 10, 1974, 487-502.

[5] T. Hellen, On the method of virtual crack extensions, *International Journal for Numerical Methods in Engineering,* 9(1), 1975, 187-207.

[6] R. Krueger, Virtual crack closure technique: History, approach, and applications, *Applied Mechanics Reviews,* 57(2), 2004, 109-143.

[7] F. Z. Li, C. F. Shih and A. Needleman, A comparison of methods for calculating energy release rate, *Engineering Fracture Mechanics,* 21, 1985, 405-421.

[8] C. Shih, B. Moran, and T. Nakamura, Energy release rate along a three-dimensional crack front in a thermally stressed body, *International Journal of Fracture,* 30(2), 1986, 79-102.

[9] H. DeLorenzi, On the energy release rate and the J-integral for 3-D crack configurations, *International Journal of Fracture,* 19, 1982, 183-193.

[10] W. Brocks and H. Yuan, Numerical investigations on the significance of J for large stable crack growth, *Engineering Fracture Mechanics,* 32(3), 1989, 459-468.

[11] W. Brocks, I. Scheider and G.-F. Geesthacht, *Numerical Aspects of the Path-Dependence of the J-integral in Incremental Plasticity*, Geesthacht*:* GKSS Forschungszentrum, 2001.

[12] G. Nikishkov and S. Atluri, Calculation of fracture mechanics parameters for an arbitrary three-dimensional crack, by the 'equivalent domain integral' method, *International Journal for Numerical Methods in Engineering,* 24(9), 1987, 1801-1821.

[13] M. Gosz and B. Moran, An interaction energy integral method for computation of mixed-mode stress intensity factors along non-planar crack fronts in three dimensions, *Engineering Fracture Mechanics,* 69(3), 2002, 299-319.

[14] T. Nakamura, Three-dimensional stress fields of elastic interface cracks, *Journal of Applied Mechanics,* 58(4), 1991, 939-946.

[15] T. Nakamura and D. M. Parks, Three-dimensional crack front fields in a thin ductile plate, *Journal of the Mechanics and Physics of Solids,* 38(6), 1990, 787-812.

[16] D. K. Yi and T. C. Wang, The effect of out-of-plane constraint on the stress fields near the front of a crack in a thin ductile plate, *International Journal of Solids and Structures,* 190, 2020, 244-257.

[17] E. Wang, W. Zhou and G. Shen, Three-dimensional finite element analysis of crack-tip fields of clamped single-edge tension specimens – Part I: Crack-tip stress fields, *Engineering Fracture Mechanics,* 116, 2014, 122-143.

[18] Y. Tkach and F. M. Burdekin, A three-dimensional analysis of fracture mechanics test pieces of different geometries – Part 1 Stress-state ahead of the crack tip, *International Journal of Pressure Vessels and Piping,* 93-94, 2012, 42-50.

[19] F. Yusof and J. W. Hancock, In-plane and out-of-plane constraint effects in three-dimensional elastic perfectly-plastic crack tip fields, *Proceedings of the 11th International Conference on Fracture*, Turin, 2005, 1-6.

[20] J. R. Rice, Mechanics of crack tip deformation and extension by fatigue, in *Fatigue Crack Propagation*, J. Grosskreutz, Ed. West Conshohocken, PA: ASTM International, 1967, 247-311.

[21] J. R. Rice, Mathematical analysis in the mechanics of fracture, in *Fracture: An Advanced Treatise,* H. Liebowitc, Ed. NY, USA: Academic Press, 2, 1968, 191-311.

[22] ABAQUS, *ABAQUS v6.12 User's Manual*, Rhode Island: Dassault Systèmes Simulia Corp., 2012.

[23] F. Yusof, *Three-Dimensional Constraint Based Fracture Mechanics,* Ph.D. Dissertation, Dept. Mech. Eng., Glasgow Univ., Glascow, 2006