ARQII Publication
APPLICATIONS OF
MODELLING AND SIMULATION
http://arqiipubl.com/ams eISSN 2600-8084 VOL 10, 2026, 98–107

LMI-based Controller Design for Leaky-integrator MIMO Systems

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Submitted 20 December 2025; Revised 27 February 2026; Accepted 04 March 2026; Available online 16 March 2026.

Abstract: Leaky-integrator systems have attracted attention due to their capability of capturing the gradual loss of the system states. Thanks to the advantages of taking into consideration the leaky effect on gradual loss of systems states, leaky-integrator systems have been applied in biological modeling and electronic circuit design. However, robust control design conditions for such systems are insufficiently developed. In the existing literature, global asymptotic stability analysis has been conducted on leaky-integrator systems, however, such analysis neglects the effects of the inputs and is based solely on single-input single-output systems. The stability analysis for leaky-integrator multi-input multi-output (MIMO) systems is yet to be developed. This paper proposes a controller design method based on Linear Matrix Inequality (LMI) for a class of leaky-integrator MIMO systems. Compared to the existing literature, controllers that guarantee incremental input-to-state stability for leaky-integrator systems are established. In this paper, a set of linear matrix inequalities regarding the controller design conditions is derived based on the incremental input-to-state stability. An observer structure for the considered leaky-integrator MIMO systems is also presented, along with the observer design conditions. A numerical simulation example showcases the effectiveness of the proposed approach. The simulation results demonstrate that the developed LMI conditions are applicable for a class of leaky-integrator MIMO systems. With the proposed controller and observer design conditions, the leaky-integrator MIMO system can be stabilized and achieve the control goal of trajectory tracking.

Keywords: LMIs; δISS; Leaky-integrator; MIMO systems.

1. INTRODUCTION

In numerous engineering applications, a phenomenon named *leaky-effect* frequently appears. The *leaky-effect* showcases the gradual loss of system state values over time. For instance, in the RC capacitor application, the leaky effect leads to an increasing leakage current in the circuit. In the field of neural network (NN), there are multiple NN models considering the leaky effect such as leaky-integrator echo state network [1]. Neglecting the *leaky-effect* can lead to model mismatch or control failure. In the application of RC circuit, neglecting the increasing leakage current possibly causes circuit overload and further results in system malfunctioning [2]. Taking the leaky effect into consideration is crucial for an accurate system dynamics description.

To represent the leaky effect in system dynamics, the leaky-integrator structure is used. The structure of leaky-integrator systems contains both a leaky part, which accounts for the leaky-effect, and an integration part that shows the state evolution over time [1]. Compared to the standard nonlinear systems, leaky-integrator systems are advantageous on both tracking the steady state value and describing the transient state dynamics. Due to its benefits on representing the leaky-effect, leaky-integrator systems have been employed in multiple engineering applications such as leaky-integrator and fire model [3], leaky-integrator neurons in neural network [4] and circuit design [5].

Analyzing multi-input multi-output (MIMO) systems is critical for optimizing system design and accounting for real-world physical limitations [6]. In practice, many engineering systems exhibit inherently coupled state variables. Considering this coupling effect is essential for accurate modeling and optimized control for the system [7]. For example, for the quadruple tank system, two pump rates serve as the inputs, both of which influence the output water tank levels differently [8]. To accurately represent the coupling effects, it is imperative to include all the input and output variables when formulating the system model.

In the existing literature, there are very few researches on the stability of leaky-integrator MIMO systems. In [9], a Global Asymptotic Stability (GAS) criterion is proposed for the leaky-integrator echo state network system. In [10], an incremental input-to-state stability (δISS) criterion is developed for leaky-integrator echo state network systems. Compared to the conventional GAS, δISS is more strict as it implies common stability properties such as GAS of the equilibrium and Input-to-State Stability (ISS) [11]. δISS entails that, asymptotically, the state trajectories are decided solely by the applied inputs instead of the initial conditions. However, both [9] and [10] focus solely on the single input single output leaky-integrator system. The stability criterion for leaky-integrator MIMO systems is yet to be developed and implemented.

The contribution of the current paper lies in the development of a LMI-based controller design method for leaky-integrator MIMO systems based on δISS. Compared to [12], the LMI-based controller method is further extended to leaky-integrator MIMO systems.

The remaining paper is organized as follows. Section 2 describes some preliminary concepts and definitions of δISS. The investigated leaky-integrator model is presented in detail. In section 3, the δISS stability criterion is derived. In Section 4, the LMI-based controller design conditions and the observer design conditions for the leaky-integrator MIMO system are presented. Section 5 shows a simulation example of a leaky-integrator MIMO system employing the developed controller method. The conclusion is drawn in Section 6.

Throughout the paper, for a given matrix X, the notation $X^\top$, X^{-1}, $\|X\|$ denotes its transpose, its inverse and its norm, respectively. The matrix I_n represents a $n \times n$ identity matrix. The notation $X \succ 0$ (respectively $X \prec 0$) indicates that the matrix X is strictly positive (negative) definite. For LMIs, the symbol $\star$ presents the symmetrical terms.

2. PROBLEM STATEMENT

This section introduces the leaky-integrator system formulation, together with some preliminary concepts that form the theoretical foundation of the proposed approach. Leaky-integrator systems are a class of dynamical systems that capture the gradual dissipation of system states over time. This gradual loss of system state values needs to be addressed in the system model to account for an accurate system dynamics representation and for subsequent controller and observer design conditions. In this section, a leaky-integrator MIMO system model is derived from the existing literature. Besides, to facilitate the subsequent stability analysis and controller design on the investigated system, the concepts of incremental input-to-state stability and mean value theorem are introduced.

2.1 Leaky-integrator System Model

With the applications in numerous applications such as MOSFET [13], leaky-integrator systems have been researched attentively in the past decades. In [14], a leaky-integrator echo state network (ESN) system model is proposed as

$$\begin{aligned} x(k+1) &= (1-\Delta r)x(k) + \Delta r f(Wx(k) + W_{in}u(k) + W_{fb}y(k)), \\ y(k) &= W_{out}x(k), \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ denotes the network states, $u \in \mathbb{R}^m$ is the network input vector and $y \in \mathbb{R}^p$ indicates the network output with the discrete-time index $k \in \mathbb{Z}_{\geq 0}$. The term Δr (denoted by γ in [14]) represents the system leaky rate with $\Delta r \in (0,1]$. The function $f(\cdot)$ is assumed to be a Lipschitz continuous differentiable nonlinear function, normally chosen as $\tanh(\cdot)$ in the leaky integrator ESN system.

The matrices $W \in \mathbb{R}^{n\times n}$, $W_{in} \in \mathbb{R}^{n\times m}$, $W_{fb} \in \mathbb{R}^{n\times p}$, $W_{out} \in \mathbb{R}^{p\times n}$ denote the connection weights. In [14], the feedback connection from the output is not considered, i.e., $W_{fb} = 0$. To simplify the notation, the classic notation A, B, C for the system matrices are adopted with respect to (1). Thus, a leaky-integrator nonlinear system with $W_{fb} = 0$ is modeled as

$$\begin{aligned} x(k+1) &= (1-\Delta r)x(k) + \Delta r f(Ax(k) + Bu(k)), \\ y(k) &= Cx(k), \end{aligned} \tag{2}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ represent the leaky-integrator system matrices. Considering the Multi-Input Multi-Output (MIMO) case, it is assumed that $m > 1$ and $p > 1$.

2.2 Preliminary Concepts

This section presents several important preliminary concepts used to derive the proposed results.

Consider a discrete-time nonlinear system as

$$x(k+1) = f(x(k), u(k)), \tag{3}$$

with $f : \mathbb{X} \times \mathbb{U} \to \mathbb{X}, \mathbb{X} \subseteq \mathbb{R}^n, \mathbb{U} \subseteq \mathbb{R}^m$. Similar to [15], the notation $x(k, x_0, \mathbf{u})$ indicates a solution of system (3) at sampling time k with an initial state $x_0 \in \mathbb{X}$ and input sequence $\mathbf{u} = (u_0, \ldots, u_{k-1}) \subseteq \mathscr{U}$, where $u_j \in \mathbb{U}$, with $j \in \{0, 1, \ldots, k-1\}$.

Definition 1 *(δISS [15]) A system (3) is called incrementally input-to-state stable if there exist a function $\gamma \in \mathscr{K}_\infty$ and a function $\beta \in \mathscr{KL}$ such that for any sampling time $k \in \mathbb{Z}_{\geq 0}$, any initial states $x_{0_1}, x_{0_2} \in \mathbb{X}$ and any couple of input sequences $\mathbf{u_1}, \mathbf{u_2} \subseteq \mathscr{U}$, the following expression holds*

$$\|x(k, x_{0_1}, \mathbf{u_1}) - x(k, x_{0_2}, \mathbf{u_2})\| \leq \beta(\|x_{0_1} - x_{0_2}\|, k) + \gamma(\|\mathbf{u_1} - \mathbf{u_2}\|_\infty).$$

Theorem 1 *(δISS Lyapunov criterion [16]) For a positive definite function V, if there exist a $\mathscr{K}$ function α_a and $\mathscr{K}_\infty$ functions α_1, α_2, α_3 such that*

$$\begin{aligned}\alpha_1(\|x_1-x_2\|) \leq V(x_1,x_2) \leq \alpha_2(\|x_1-x_2\|),\\ V(f(x_1,u_1),f(x_2,u_2)) - V(x_1,x_2) \leq -\alpha_3(\|x_1-x_2\|)+\alpha_a(\|u_1-u_2\|),\end{aligned} \tag{4}$$

for all x_1, $x_2 \in \mathbb{X}$, u_1, $u_2 \in \mathbb{U}$, then, the considered system is called incremental Input-to-State-Stable, with V defined as the δISS Lyapunov function.

Theorem 2 *(Mean-Value Theorem [9]) Assume f is a real-valued differentiable function defined on an open set $\mathscr{L} \subseteq \mathbb{R}^n$. For two points $a,b \in \mathscr{L}$, let $g_{a,b}$ denote the line segment that connects them. If $g_{a,b} \subseteq \mathscr{L}$, then there exists a point $c \in g_{a,b}$ such that*

$$\nabla f(c)\cdot(b-a) = f(b)-f(a), \tag{5}$$

with $\nabla f(c)$ defined as the gradient of f at point c.

3. STABILITY CRITERION OF A LEAKY INTEGRATOR NONLINEAR SYSTEM

This section derives the δISS stability criterion for a class of discrete-time leaky-integrator nonlinear systems. For a leaky-integrator system (2), the following theorem is proposed for verifying the δISS stability criterion.

Theorem 3 *Considering the leaky integrator system (2), if there exist a nonsingular matrix $G \in \mathbb{R}^{n\times n}$, a symmetric matrix $P = P^\top \succ 0$, $P \in \mathbb{R}^{n\times n}$ such that*

$$\begin{bmatrix} P-G-G^\top & 0 & ((1-\Delta r)G+\Delta rLAG)^\top & G^\top \\ * & -I & I & 0 \\ * & * & -P & 0 \\ * & * & * & -I \end{bmatrix} \prec 0, \tag{6}$$

where $L = diag(l_{p1},\ldots,l_{pn})$, with l_{pi} defined as the component-wise derivative term for $i \in \{1,\ldots,n\}$ in (5), then the corresponding leaky-integrator system (2) is δISS.

Proof Considering two different states $x_1,x_2 \in \mathbb{X}$ and $u_1,u_2 \in \mathbb{U}$ of the investigated system (2), it can be written that

$$\begin{aligned} x_1(k+1) = (1-\Delta r)x_1(k) + \Delta r f(Ax_1(k)+Bu_1(k)),\\ x_2(k+1) = (1-\Delta r)x_2(k) + \Delta r f(Ax_2(k)+Bu_2(k)).\end{aligned} \tag{7}$$

Defining $z(k) = x_1(k)-x_2(k)$ and $\delta u(k) = u_1(k)-u_2(k)$, it can be obtained that

$$z(k+1) = (1-\Delta r)z(k) + \Delta r(f(Ax_1(k)+Bu_1(k)) - f(Ax_2(k)+Bu_2(k))). \tag{8}$$

Assuming that the points $Ax_1(k)+Bu_1(k), Ax_2(k)+Bu_2(k)$ and the segment in between are on an open set where f is continuously differentiable, $\forall k > 0$ and applying *component-wise* the Mean-Value Theorem from Definition 2, it leads to

$$z(k+1) = (1-\Delta r)z(k) + \Delta r L(Ax_1(k)+Bu_1(k)-Ax_2(k)-Bu_2(k)), \tag{9}$$

where $L = L(z(k),\delta u(k))$, with $L = \mathrm{diag}(l_{p1},\ldots,l_{pn})$, see [9]. Since $z(k)$ and $\delta u(k)$ are defined previously, it can be simplified as

$$z(k+1) = (1-\Delta r)z(k) + \Delta rLAz(k) + \Delta rLB\delta u(k). \tag{10}$$

Define $V(k) = z(k)^\top \tilde{P} z(k)$ as a Lyapunov candidate function. Choosing $\alpha_1(z(k)) = \lambda_{\min}(\tilde{P})\|z(k)\|^2$ and $\alpha_2(z(k)) = \lambda_{\max}(\tilde{P})\|z(k)\|^2$, it follows that

$$\lambda_{\min}(\tilde{P})\|z(k)\|^2 \leq V(k) \leq \lambda_{\max}(\tilde{P})\|z(k)\|^2, \tag{11}$$

where $\lambda_{\min}$ and $\lambda_{\max}$ are the minimal and maximal eigenvalues of matrix $\tilde{P}$, respectively. Thus, the first condition in (4) has now been proved. In order to prove the second condition in (4), compute $V(k+1)-V(k)$ as

$$V(k+1)-V(k) = \|(1-\Delta r)z(k) + \Delta rLAz(k) + \Delta rLB\delta u(k)\|_{\tilde{P}}^2 - \|z(k)\|_{\tilde{P}}^2. \tag{12}$$

To simplify the notation in (12), define

$$a_1 = (1-\Delta r)I, \tag{13a}$$
$$b_1 = \Delta rLA, \tag{13b}$$
$$c_1 = \Delta rLB. \tag{13c}$$

Therefore, (12) is now

$$V(k+1)-V(k)=\|(a_1+b_1)z(k)+c_1\delta u(k)\|_{\tilde{P}}^2-\|z(k)\|_{\tilde{P}}^2. \tag{14}$$

Define an additional variable $w(k)=c_1\delta u(k)$, this yields that

$$V(k+1)-V(k)=((a_1+b_1)z(k)+w(k))^\top\tilde{P}(((a_1+b_1)z(k)+w(k)))-z(k)^\top\tilde{P}z(k). \tag{15}$$

To fulfill the second condition of the δISS property in Theorem 1, choose

$$V(k+1)-V(k)\leq-\|z(k)\|^2+\|w(k)\|^2, \tag{16}$$

with $\alpha_3(z(k))=\|z(k)\|^2$ and $\alpha_a(\delta u(k))=\|c_1\delta u(k)\|^2$, where c_1 is defined in (13c). With (15) and reorganizing the elements to the left hand side, it leads to

$$\begin{bmatrix} z(k)^\top & w(k)^\top \end{bmatrix}\begin{bmatrix}\phi_1 & \phi_2\\ * & \phi_3\end{bmatrix}\begin{bmatrix} z(k)\\ w(k)\end{bmatrix}\leq 0, \tag{17}$$

where

$$\begin{aligned}\phi_1&=(a_1+b_1)^\top\tilde{P}(a_1+b_1)-\tilde{P}+I,\\ \phi_2&=(a_1+b_1)^\top\tilde{P},\\ \phi_3&=-I+\tilde{P}.\end{aligned} \tag{18}$$

This is satisfied if

$$\begin{bmatrix}\phi_1 & \phi_2\\ * & \phi_3\end{bmatrix}\prec 0. \tag{19}$$

□

Applying the definitions of ϕ_1, ϕ_2 and ϕ_3, the inequality (19) can be decomposed as

$$\begin{bmatrix}-\tilde{P} & 0\\ 0 & -I\end{bmatrix}+\begin{bmatrix}a_1+b_1 & I\\ I & 0\end{bmatrix}^\top\begin{bmatrix}\tilde{P} & 0\\ 0 & I\end{bmatrix}\begin{bmatrix}a_1+b_1 & I\\ I & 0\end{bmatrix}\prec 0. \tag{20}$$

Using the Schur Complement, it can be written as

$$\begin{bmatrix}-\tilde{P} & 0 & (a_1+b_1)^\top & I\\ * & -I & I & 0\\ * & * & -\tilde{P}^{-1} & 0\\ * & * & * & -I\end{bmatrix}\prec 0. \tag{21}$$

With a change of variable $P=\tilde{P}^{-1}$, where $P=P^\top\succ 0$, it follows that

$$\begin{bmatrix}-P^{-1} & 0 & (a_1+b_1)^\top & I\\ * & -I & I & 0\\ * & * & -P & 0\\ * & * & * & -I\end{bmatrix}\prec 0. \tag{22}$$

Pre and post multiply the LMIs with $\operatorname{diag}(G^\top,I,I,I)$ and $\operatorname{diag}(G,I,I,I)$, respectively, this leads to

$$\begin{bmatrix}-G^\top P^{-1}G & 0 & G^\top(a_1+b_1)^\top & G^\top\\ * & -I & I & 0\\ * & * & -P & 0\\ * & * & * & -I\end{bmatrix}\prec 0. \tag{23}$$

With $-G^\top P^{-1}G\preceq P-G-G^\top$, this is satisfied if

$$\begin{bmatrix}P-G-G^\top & 0 & (a_1G+b_1G)^\top & G^\top\\ * & -I & I & 0\\ * & * & -P & 0\\ * & * & * & -I\end{bmatrix}\prec 0. \tag{24}$$

Applying the definition of a_1 and b_1, the LMIs (6) can be obtained.

4. LMI-BASED CONTROLLER DESIGN

In this section, a set of LMI-based controller design conditions are derived based on the δISS criterion. Additionally, an observer structure for leaky-integrator systems is presented for state estimates.

4.1 LMI-based Control Design Condition

Based on the developed δISS stability condition, now the corresponding control design conditions can be derived. Suppose the system matrix A can be written as $A = F + QK$, where F and Q are known matrices, and K is a controller gain related term. The controller related term K can be designed with the following theorem such that the leaky-integrator system (2) is δISS.

Theorem 4 *For the leaky-integrator system (2), with $A = F + QK$, if there exist a symmetric positive definite matrix $P = P^\top \succ 0$, nonsingular matrix G, matrix H such that*

$$\begin{bmatrix} P-G-G^\top & 0 & ((1-\Delta r)G+\Delta rLFG+\Delta rLQH)^\top & G^\top \\ * & -I & I & 0 \\ * & * & -P & 0 \\ * & * & * & -I \end{bmatrix} \prec 0, \tag{25}$$

where $L = diag(l_{p1}, \ldots, l_{pn})$, if we set $K = HG^{-1}$, then the corresponding leaky integrator system is incrementally input-to-state stable.

Proof From LMIs (6), with $A = F + QK$, it can be obtained that

$$\begin{bmatrix} P-G-G^\top & 0 & ((1-\Delta r)G+\Delta rL(F+QK)G)^\top & G^\top \\ * & -I & I & 0 \\ * & * & -P & 0 \\ * & * & * & -I \end{bmatrix} \prec 0, \tag{26}$$

which further leads to

$$\begin{bmatrix} P-G-G^\top & 0 & ((1-\Delta r)G+\Delta rLFG+\Delta rLQKG)^\top & G^\top \\ * & -I & I & 0 \\ * & * & -P & 0 \\ * & * & * & -I \end{bmatrix} \prec 0. \tag{27}$$

Denoting by $K = HG^{-1}$, it confirms that the LMIs (25) can be now obtained. □

4.2 Observer Design

In many industrial applications, full access to system states are required for optimization and monitoring purposes. In the case of system states not accessible, the state estimates can be used. In this section an observer structure for the leaky-integrator system (2) is proposed and the corresponding observer gain design conditions are derived. Similarly to [12], the observer proposed for the leaky-integrator system with state x_s has the following equations

$$\begin{aligned} \hat{x}_s(k+1) &= (1-\Delta r)\hat{x}_s(k) + \Delta r f_s(A\hat{x}_s(k) + Bu_s(k)) + \tilde{L}(y_s(k) - \hat{y}_s(k)), \\ \hat{y}_s(k) &= C\hat{x}_s(k), \end{aligned} \tag{28}$$

where $\tilde{L}$ is the observer gain and $\hat{x}_s$ is the state estimate. The design of the observer gain is detailed in Theorem 5.

Theorem 5 *Considering the leaky-integrator system (2), employing the observer structure in (28), the observer gain $\tilde{L}$ can be designed with*

$$\begin{bmatrix} I & ((1-\Delta r)I + \Delta rLA - \tilde{L}C)^\top \\ \star & I \end{bmatrix} \succ 0, \tag{29}$$

such that the estimation error converges to zero as k goes to infinity.

Proof Define $e_s(k) = x_s(k) - \hat{x}_s(k)$ as the estimation error. With (2) and (28), the norm of the estimation error can be written as

$$\begin{aligned} \|e_s(k+1)\| = ||&(1-\Delta r)e_s(k) + \Delta r f_s(Ax_s(k) + Bu_s(k)) \\ &-\Delta r f_s(A\hat{x}_s(k) + Bu_s(k)) - \tilde{L}Ce_s(k))||. \end{aligned} \tag{30}$$

Applying the Mean-Value Theorem in Theorem 2 *component-wise*, the following expression can be further deduced

$$\|e_s(k+1)\| = \left\|(1-\Delta r)e_s(k) + \Delta rLAe_s(k) - \tilde{L}Ce_s(k)\right\|, \tag{31}$$

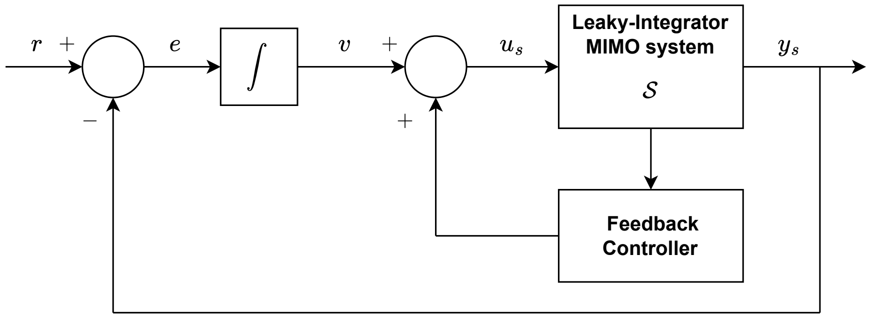

Figure 1. Leaky-integrator MIMO system state feedback control with integral action.

where $L = \text{diag}(l_1, \ldots, l_n)$, with l_d for $d \in \{1, \ldots, n\}$ denoting the component-wise derivative term of function $f_s(\cdot)$. Expression (31) further leads to

$$\|e_s(k+1)\| \le \left\|((1-\Delta r)I + \Delta r LA - \tilde{L}C)\right\| \cdot \|e_s(k)\|. \tag{32}$$

In order to ensure that the estimation error e_s converges to 0, it is expected that

$$\left\|(1-\Delta r)I + \Delta r LA - \tilde{L}C\right\| < 1. \tag{33}$$

It can be further written as

$$I - \left((1-\Delta r)I + \Delta r LA - \tilde{L}C\right)^\top \cdot I \cdot \left((1-\Delta r)I + \Delta r LA - \tilde{L}C)\right) \succ 0. \tag{34}$$

With the Schur complement, the LMIs (29) can be obtained, ensuring the convergence of $\hat{x}_s$ to x_s. □

5. SIMULATION EXAMPLE

In this section, a simulation example is presented to illustrate the effectiveness of the proposed controller design and observer design for the leaky-integrator MIMO system.

5.1 Leaky-integrator MIMO System Modelling

Consider the following leaky-integrator nonlinear system

$$\begin{aligned} x_s(k+1) &= (1-\Delta r)x_s(k) + \Delta r f_s(Ax_s(k) + Bu_s(k)), \\ y_s(k) &= Cx_s(k), \end{aligned} \tag{35}$$

with the state vector $x_s \in \mathbb{R}^2$, input vector $u_s \in \mathbb{R}^2$ and output vector $y_s \in \mathbb{R}^2$. The system matrices are

$$A = \begin{bmatrix} -0.8965 & -0.1558 \\ -0.0293 & 0.4189 \end{bmatrix}, B = \begin{bmatrix} -0.0613 & 0.1020 \\ -0.1184 & -0.0244 \end{bmatrix}, C = \begin{bmatrix} -30.8886 & -21.0622 \\ -18.4756 & 10.0423 \end{bmatrix}.$$

Define a nonlinear function $f_s(\cdot) = [f_{s_1}(\cdot)^\top \;\; f_{s_2}(\cdot)^\top]^\top$, with $f_{s_1}(\cdot) = tanh(\cdot)$ and $f_{s_2}(\cdot) = id(\cdot)$, which incorporates both the linear term and the nonlinear term for a generalized case.

5.2 State Feedback Controller Design

As one of the most common control methods, feedback control has been applied in multiple applications due to its benefits of closed-loop system error correction [17]. In this simulation example, the state feedback control strategy is adopted and the controller is designed with the derived control conditions in (25). Additionally, in multiple control problems such as trajectory tracking, establishing a zero steady state error is required. Therefore, in this simulation example an explicit integral action is incorporated with the controller structure to achieve zero steady state error.

The control law for the state feedback control is defined as

$$u_s(k) = K_s x_s(k) + v(k). \tag{36}$$

In Figure 1, the closed-loop system block diagram employing state feedback control strategy with integral action is presented.

In order to achieve trajectory tracking, an explicit integrator is incorporated with the state feedback control strategy. Similar to [10], the integrator has the following dynamics

$$\begin{aligned} \eta(k+1) &= \eta(k) + e(k), \\ v(k) &= M(\eta(k) + e(k)), \end{aligned} \tag{37}$$

where η is the integrator internal state and M denotes the integrator gain matrix yet to be designed. The control signal u_s for the leaky-integrator MIMO system is now

$$\begin{aligned} u_s(k) &= K_s x_s(k) + M(\eta(k) + e(k)), \\ e(k) &= r(k) - W_{out} x_s(k). \end{aligned} \tag{38}$$

It can be proved that the closed-loop system can be written in the form of (2) and its control gain K_s can be designed with the proposed Theorem 4. Define an augmented system state as

$$x_a(k) = \begin{bmatrix} x_s(k) \\ \eta(k) \end{bmatrix},$$

including both the state and integrator dynamics. With (35) and (37), the augmented system dynamics can be represented by

$$x_a(k+1) = A_n x_a(k) + \Delta r f(A_a x_a(k) + B_a u(k)), \tag{39}$$

where $f(\cdot) = [f_s(\cdot) \;\; id(\cdot)]^\top$ with $id(\cdot)$ denoting the identity function. Here, $u(k) = r(k)$ is the reference signal. The matrices A_a, A_n, B_a are derived with equations (35), (37), and (38) for $\eta(k+1)$, $x_s(k+1)$ respectively, similar as in [12]

$$A_a = \begin{bmatrix} A + BK_s - BMC & BM \\ -C/\Delta r & 1/\Delta r \end{bmatrix}, B_a = \begin{bmatrix} BM \\ 1/\Delta r \end{bmatrix}, A_n = \begin{bmatrix} (1-\Delta r) I_n & 0_{n,1} \\ 0_{1,n} & 0 \end{bmatrix}.$$

It can be noticed that the matrix A_a can be written in the form of $A_a = F + QK$ as in Theorem 4, with

$$F = \begin{bmatrix} A & 0_{n,1} \\ -C/\Delta r & 1/\Delta r \end{bmatrix}, \quad G = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad K = \begin{bmatrix} K_s - MC & M \end{bmatrix}.$$

To obtain the controller gain related term K_s and the integrator gain M, a matrix E is now defined with

$$E = \begin{bmatrix} I_n & 0_{n,1} \\ -C & 1 \end{bmatrix}. \tag{40}$$

Therefore, the controller gain related term K_s and the integrator gain M can be designed with

$$\begin{bmatrix} K_s & M \end{bmatrix} = KE^{-1}, \tag{41}$$

such that the closed-loop system achieves trajectory tracking with respect to the reference signal r, and fulfills the δISS property.

5.3 Result Analysis

In this section, implementing the controller and observer using the developed Theorem 4 and Theorem 5 to the leaky-integrator system (35), the output response for the leaky-integrator MIMO system is presented in Figure 2.

It can be observed from the Figure 2 that, with the implemented controller design conditions and the explicit integral action, the outputs of the leaky-integrator system are able to track the input trajectories. Despite the fluctuation in the initial transient stage, both system outputs (in black line) and estimates (in red line) converge to the reference trajectory (in blue dotted line). In the MIMO scenario, given two inputs, the leaky-integrator system outputs y_{s1}, y_{s2} follows the reference inputs r_1, r_2 correspondingly, resulting in the success of trajectory tracking. Besides, with the introduced observer design conditions, the estimated system outputs $\hat{y}_1$, $\hat{y}_2$ converge to the real system outputs y_{s1}, y_{s2} with a minimized estimation error applying the observer gain design condition (29). The obtained control gain from Theorem 4 by using the control strategy in Section 5.2 and the obtained observer gain by Theorem 5 are

$$K = \begin{bmatrix} -1.8523 & 2.7856 \\ 7.7845 & 3.2023 \end{bmatrix}, \quad L = \begin{bmatrix} 0.0139 & 0.0151 \\ -0.0122 & 0.0219 \end{bmatrix}.$$

To investigate the robustness of the derived control method for systems with a different leaky rate, the leaky rate Δr is now decreased from 0.9 to 0.7. Figure 3 illustrates the system output response regarding a different system leaky rate $\Delta r = 0.7$. Compared to Figure 2, the output response in Figure 3 experiences a larger fluctuation at the initial transient stage when the reference signal changes. In spite of the initial transient fluctuations, the system outputs converge to the reference trajectory. Both of the two outputs are capable of tracking each reference trajectory individually, demonstrating the effectiveness of the derived controller design conditions (25). To evaluate the tracking performance in Figure 2 and Figure 3, the Root Mean Square Error (RMSE) is calculated with respect to the reference trajectory. In Figure 2, the RMSE for the two outputs are 1.5846 and 1.5320, while in Figure 3, the RMSE for both outputs are 1.8107 and 1.7297. Compared to the case of $\Delta r = 0.7$, when $\Delta r = 0.9$, the RMSE has a reduction percentage around 12.5% and 11.4% for both outputs, respectively. To evaluate the observer response, the estimation errors between x_s and $\hat{x}_s$ are plotted in Figure 4 and Figure 5. It can be observed that with the observer gain designed by the proposed LMIs (29), the estimation errors in both cases converge to zero, hence showing that the estimated states converge to the real system states.

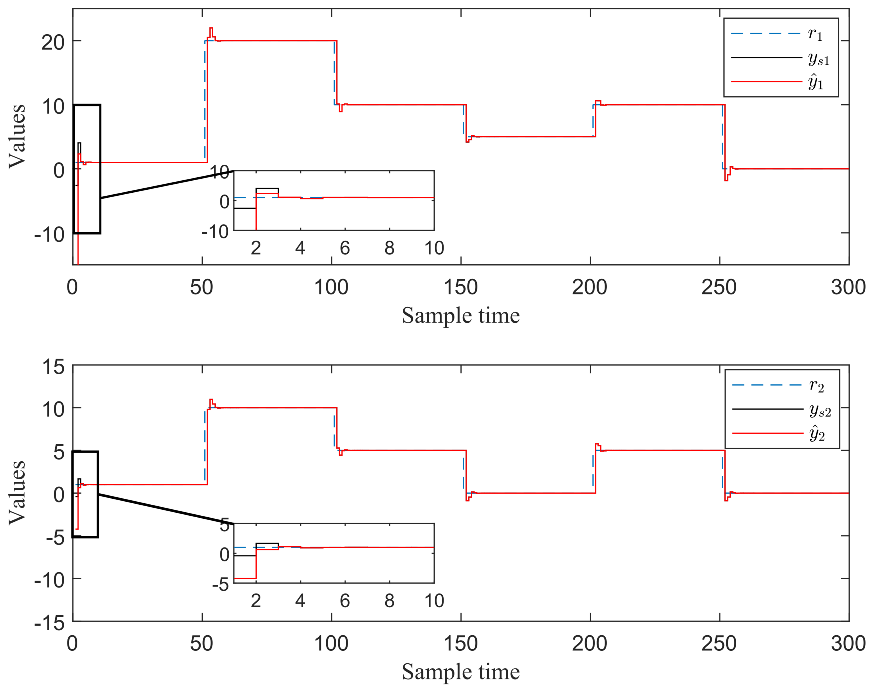

Figure 2. Leaky-integrator MIMO system output response with $\Delta r = 0.9$.

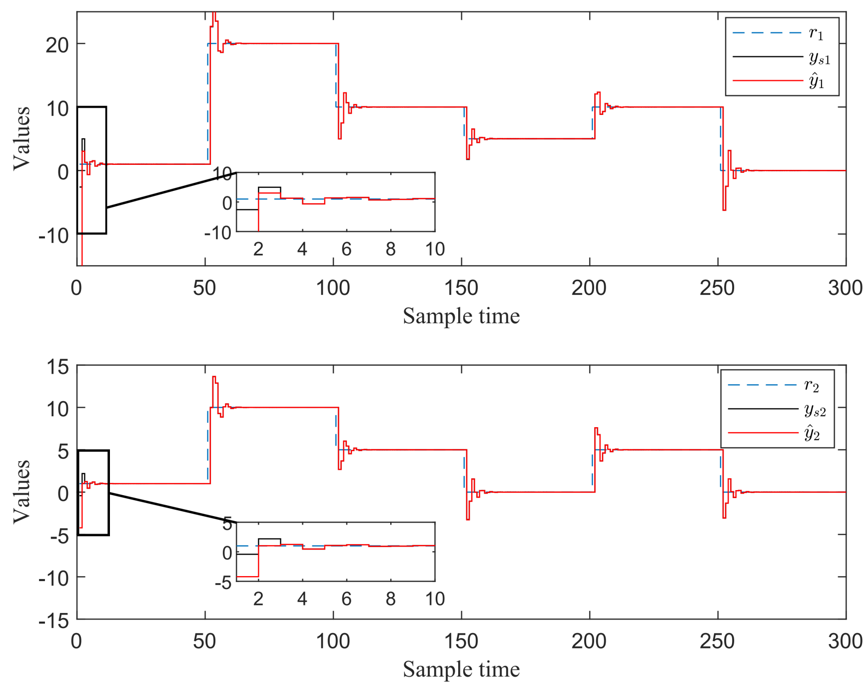

Figure 3. Leaky-integrator MIMO system output response with $\Delta r = 0.7$.

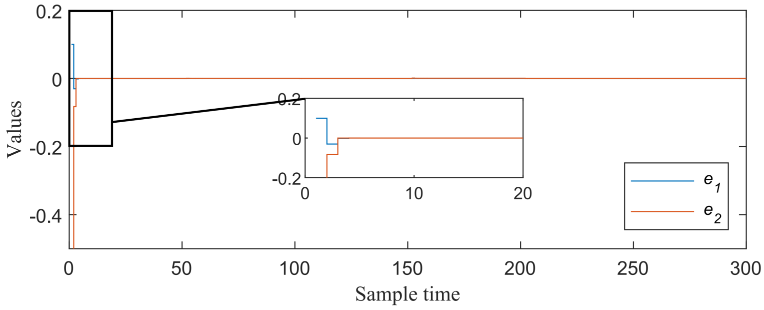

Figure 4. Estimation error response with $\Delta r = 0.9$.

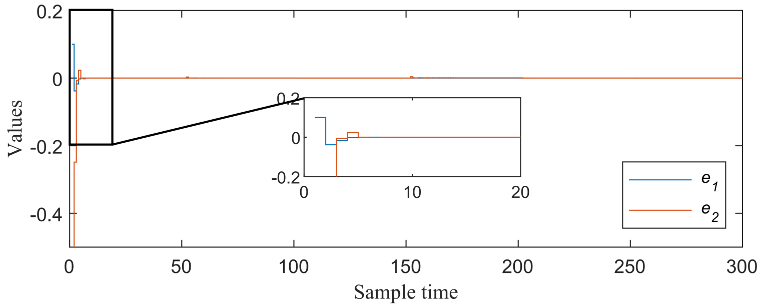

Figure 5. Estimation error response with $\Delta r = 0.7$.

6. CONCLUSION

This paper introduces a LMI-based controller design method for leaky-integrator MIMO systems. The controller design conditions are derived based on the incremental Input-to-State Stability criterion. The corresponding observer structure for the leaky-integrator MIMO systems is proposed, as well as the observer design conditions derived. The simulation example illustrated the effectiveness of the derived controller and observer for the leaky-integrator MIMO systems. The future work concerns the extension on implementing the algorithm on an existing MIMO benchmark.

ACKNOWLEDGEMENT AND FUNDING

This work was supported by the French Government through the France 2030 Program, within the OI H-Code and the STIC Doctoral School, Université Paris-Saclay under Grant ANR-11-IDEX-0003. The work of Cristina Stoica was supported in part by the Regional Program STIC-AmSud under Grant STIC-AMSUD 23-STIC14, and in part by the Agence Nationale de Recherche France under Grant ANR-21-CE48-0003.

DECLARATION OF CONFLICTING INTERESTS

The authors declare no potential conflicts of interest with respect to the research and publication of this article.

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